Robust Estimates of Climate Change and a Generalization of Structural Stability

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Thursday, 19 April 2007, EGU, Vienna
(1) Motivation: IPCC and uncertainty range.
(2) Structural stability: the topological point of view.
(3) Random Dynamical Systems (RDS), definitions and examples.
(4) Stochastic equivalence, and a robust classification.
(5) Concluding remarks.
Motivation: IPCC and uncertainty range - Is it merely an engineering problem or an intrinsic one?

- Stubborn persistence in range of uncertainties
- Sensitivity to parameters and parametrizations
- Nonlinear and multi-scale effects

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**Topological conjugacy**: Flow $A \approx Flow\ B$ if there exists a homeomorphism $h$ such that

$$h^{-1} \circ (\text{Flow } A) \circ h = \text{Flow } B.$$ 

**Which systems are structurally stable?** **Hyperbolic ones!** (i.e. with invariant hyperbolic sets - e.g. Smale’s horseshoes). Unfortunately this is **not a generic situation**...
The three towers of differentiable dynamics:
The Lorenz system is structurally unstable! (Guckenheimer, 1976; Guckenheimer & Williams, 1979)

But the coarse-grained structure of the Lorenz attractor is robust.
One would like to consider invariant (SRB) measures, instead of topological structure.
This requires a much more general framework than provided by dynamical systems theory.
Random Dynamical Systems - RDS theory

This theory is a combination of measure (probability) theory and dynamical systems initiated by the “Bremen group” (L.Arnold, 1998). It allows one to treat Stochastic Differential Equations (SDEs), and more general systems driven by some “noise”, as flows.

Setting:

(i) A phase space $X$. **Example:** $\mathbb{R}^n$.

(ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example:** The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P} = \gamma$.

(iii) A model of the noise $\theta(t) : \Omega \to \Omega$ that preserves the measure $\mathbb{P}$, i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; $\theta$ is called the driving system. **Example:** $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$; it starts the noise at $s$ instead of $t = 0$.

(iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \to X$ with the cocycle property. **Example:** The solution of an SDE.
Random Dynamical Systems - A geometric view of SDEs

\[ \varphi \] is a random dynamical system (RDS)

\[ \Theta(t)(x, \omega) = (\theta(t)\omega, \varphi(t, \omega)x) \] is a flow on the bundle
A random attractor $A(\omega)$ is both invariant and "pullback" attracting:

(a) **Invariant:** $\varphi(t, \omega) A(\omega) = A(\theta(t) \omega)$.

(b) **Attracting:** $\forall B \subset X, \lim_{t \to \infty} \text{dist} (\varphi(t, \theta(-t) \omega) B, A(\omega)) = 0$ a.s.

*Pullback attraction to $A(\omega)$*
Pitchfork with additive noise (H. Crauel & F. Flandoli, 1998):

\[ dX = (\mu X - X^3)dt + \varepsilon dW_t, \]

This system has a unique solution \( \varphi(t, \omega)x \) for all initial data \( x \), \( t \geq 0 \), and realization \( \omega \) of \( W(t) \). One defines the family of measure-preserving maps \( (\theta(t))_{t \in \mathbb{R}} \) such that, on the Wiener space \( (\Omega, \gamma) \), \( W_t(\theta(s)\omega) = W_{t+s}(\omega) - W_s(\omega) \).

The SDE then generates a RDS \( (\theta, \varphi) \).

**Theorem (H. Crauel & F. Flandoli, 1998)**

For all \( \mu \in \mathbb{R} \) and all \( \varepsilon > 0 \), the system has a random attractor \( A_{\mu,\varepsilon}(\omega) = A(\omega) = \{a(\omega)\} \), where \( a(\omega) \) is a single point, distributed with density \( p_{\mu,\varepsilon}(x) = Ce^{(\mu x^2 - x^4/2)/\varepsilon^2} \). The measure \( \nu(\omega) = \delta_{a(\omega)} \) depends on the realization \( \omega \) and is an invariant measure. The density has one peak at 0 for \( \mu \leq 0 \) and two peaks at \( \pm \sqrt{\mu} \) for \( \mu > 0 \).
Multiplicative Ergodic Theorem (Oseledec)

Provides a **spectral theory** for linear cocycles in RDS theory using **Lyapunov exponents** and, as a by-product of this theory, a version of the Hartman-Grobman theorem. It is at the heart of RDS theory and of the "classification" of RDSs.
Some nuances w.r.t. classical bifurcation theory

- **D-bifurcation** (dynamical) - bifurcation of the invariant measure; the Lyapunov exponent changes its sign.
- **P-bifurcation** (phenomenological) - qualitative change in the density.

Pitchfork with multiplicative noise:

\[
dX = (\mu X - X^3)dt + \varepsilon X \circ dW_t
\]

This SDE gives rise to the RDS \((\theta, \varphi_\mu)\):

\[
\varphi_\mu(t, \omega) x = \frac{x \exp(\mu t + \varepsilon W_t(\omega))}{\left(1 + 2x^2 \int_0^t \exp(2\mu s + 2\varepsilon W_s(\omega)) \, ds\right)^{1/2}}
\]
Random Dynamical Systems - Pitchfork with multiplicative noise
A tool for classification: stochastic conjugacy

- **Stochastic conjugacy**: two cocycles \( \varphi_1(t, \omega) \) and \( \varphi_2(t, \omega) \) are conjugated iff there exists a **random homeomorphism** \( h \in \text{Homeo}(X) \) and an invariant set \( \tilde{\Omega} \) of full \( \mathbb{P} \)-measure (w.r.t. \( \theta \)) such that \( h(\omega)(0) = 0 \) and:

\[
\varphi_1(t, \omega) = h(\theta(t)\omega)^{-1} \circ \varphi_2(t, \omega) \circ h(\omega); \quad (1)
\]

\( h \) is also called **cohomology** of \( \varphi_1 \) and \( \varphi_2 \). It is a **random change of variables**!

- **Motivation**: We would like to measure quantitatively the difference between **climate models**. As the noise variance tends to zero and/or the parametrizations are switched off, one recovers the structural instability, as a “granularity" of model space. For **nonzero variance**, the random attractor \( \mathcal{A}(\omega) \) associated with several GCMs might fall into **larger** and **larger** classes as the noise level increases.

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Stochastic equivalence - Could noise help the classification?

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Concluding remarks

- **Difference between models:**

\[
\begin{align*}
(GCM - \text{team})_1 : \quad dU &= f_1(U)dt + \sigma_1(x, U)dW_t \\
(GCM - \text{team})_2 : \quad dU &= f_2(U)dt + \sigma_2(x, U)dW_t
\end{align*}
\]

Under which conditions on \( f_1 - f_2 \) and \( \sigma_1 - \sigma_2 \) will \( A_1(\omega) \approx A_2(\omega) \) hold?

- **Increase in resolution:**
  - Let \( k \) denote the GCM resolution \( dU = f(U, \theta(t)\omega, k)dt \).
  - One would like to study the behavior of \( A_k(\omega) \) as \( k \to 0 \).

- **Model validation with data:**
  - Joint analysis of model simulations and observational data sets
  - Parameter estimation, based on data assimilation methods (sequential, variational)