A Nonlinear Stochastic Model of the El Niño–Southern Oscillation: Random Dynamical Systems and Predictability

E. Simonnet[1], M. D. Chekroun[2] and M. Ghil[3]

[1]Institut Non-Linéaire de Nice, UMR 6618, CNRS-UNSA, Valbonne, France
[2]Environmental Research and Teaching Institute, École Normale Supérieure, Paris, France
[3]Atmospheric and Oceanic Sciences Dept. and IGPP, UCLA, Los Angeles, USA.

1. Motivation

- What is the effect of stochastic (atmospheric?) perturbations on models of ENSO variability?
- **Aim:** Use of random dynamical system (RDS) theory to assess the dynamical effects of noise → "PDF dynamics".

2. Random Dynamical Systems (RDSs)

- We have a model of the noise $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ that is parametrized by time. The parametrization of realizations ω is provided by a **driving system** θ which is an **ergodic one-parameter group**.
- The dynamics is viewed on a "phase space × probability space," $X \times \Omega$, called the **bundle**, and the cocycle property enables one to treat trajectories as *flows* on this bundle.
- A path of the stochastic process corresponds to a selection of points in each fiber of the resulting bundle. Fibers are "glued together" by noise. The cocycle, also called RDS, provides a fiber-by-fiber view of the dynamics.

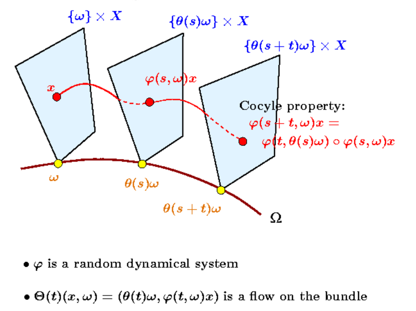


Figure 1: The bundle and the cocycle: the stochastic dynamics as a flow.

3. Random attractors and invariant measures

- We are looking for measures μ on $\Omega \times X$ that are invariant w.r.t. the dynamics. Central to our approach is the concept of disintegration μ_ω of μ. Mathematically, it is given by $\mu(B) = \int_\Omega \mu_\omega(B_\omega)\mathbb{P}(d\omega)$, with B_ω the intersection of a measurable set B of $X \times \Omega$ with an ω-fiber.
Physically? Let's see.
- μ_ω is numerically computable for each fiber. It is supported by a geometric object: the random attractor.
- The random attractor, $\mathcal{A}(\omega)$ in Fig. 2, involves pullback attraction: we look at the phase-space location at time t starting several experiments far enough in the past and *for the same realization*. Hence we assess the "attracting regime" at time t.
- The sample measure μ_ω evolves with time, $\mu_\omega \mapsto \mu_{\theta(t)\omega}$, and corresponds to the frozen statistics at time t: for each piece of $\mathcal{A}(\omega)$, it gives the probability to end up on that piece.

Schematic view of the random attractor's life

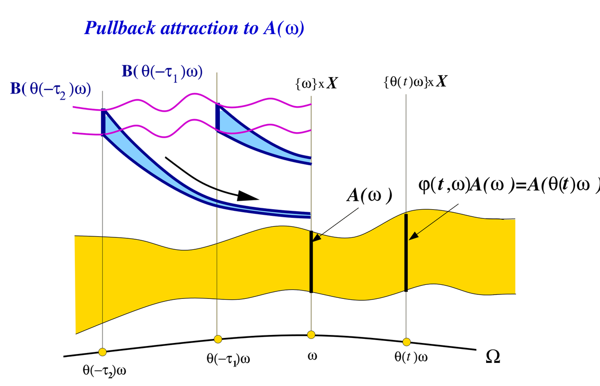


Figure 2: The curved arrow depicts the pullback attraction.

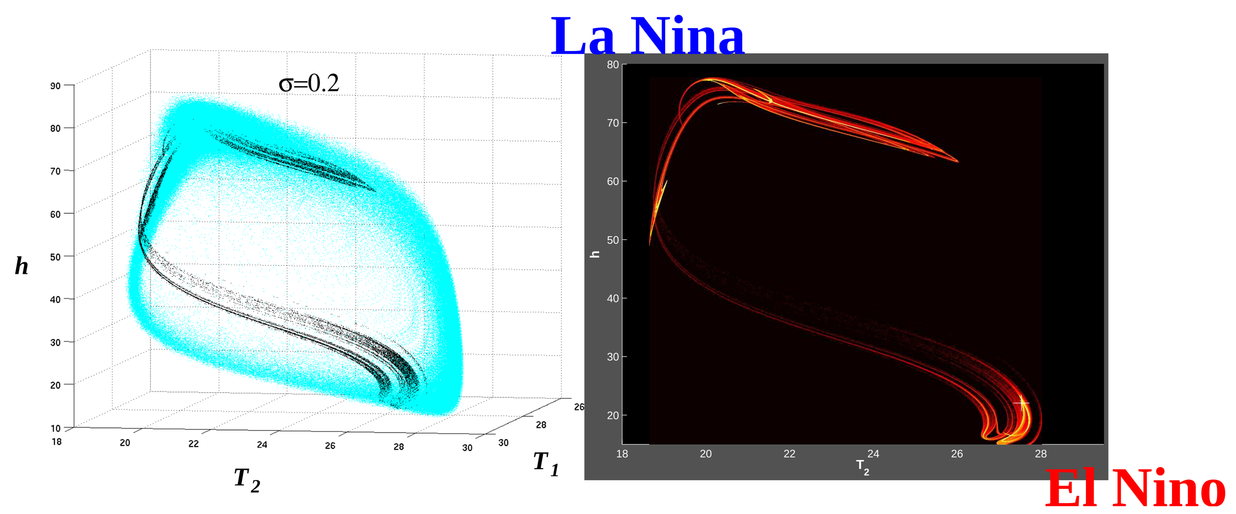
4. Application to an ENSO model

- **Model**:
Low-order, coupled tropical-atmosphere–ocean model of ENSO (Timmermann & Jin, *GRL*, 2002; hereafter **TJ model**). Three variables: thermocline depth anomaly h, and SSTs T_1 and T_2 in the western and eastern basin.

$$\begin{aligned}\dot{T}_1 &= -\alpha(T_1 - T_r) - \tfrac{2\epsilon u}{L}(T_2 - T_1),\\ \dot{T}_2 &= -\alpha(T_2 - T_r) - \tfrac{w}{H_m}(T_2 - T_{sub}),\\ \dot{h} &= r(-h - bL\tau/2),\end{aligned}$$

$$\begin{aligned}T_{sub} &= T_r - \tfrac{T_r - T_{r0}}{2}[1 - \tanh(H + h_2 - z_0)/h^*]\\ \tau &= \tfrac{a}{\beta}(T_1 - T_2)[\xi_t - 1].\end{aligned}$$

The quantities are τ, the wind stress anomalies, equatorial upwelling $w = -\beta\tau/H_m$, zonal advection $u = \beta L\tau/2$, and subsurface temperature T_{sub}. Wind stress bursts are modeled as white noise ξ_t of variance σ and ϵ measures the strength of the zonal advection. We refer to **TJ** for the other parameters.
- **Noise effects**:
  - **Traditionnally**: noise effects on deterministic dynamics are studied by forward integration (in blue in Fig. 3). The system being nonautonomous, "forward-in-time" dynamics is ill-defined;
  static view of the statistics: PDF.
  - **RDS approach**: Pullback dynamics is relevant;
  dynamical view of the statistics: μ_ω.
  The probability to be in a particular location at time t is given by the measure μ_ω supported by the random attractor (in red on Fig. 3).



μ_ω**(in red) is in log-scale**

Figure 3: Classical forward-in-time integration of the stochastic TJ model (left).
The corresponding random attractor (right) with El Niño/La Niña phase-space areas.

- The random attractor of Fig. 3 corresponds to an El Niño episode: the maximum of μ_ω (white 'plus' sign) is located at low h and high T_2.

5. Deterministic case: Shilnikov route to chaos

- Complex deterministic dynamics: **Hopf bifurcation** and **Shilnikov horseshoes** associated with **multi-pulse homoclinic orbits**, as ϵ varies. Interdecadal chaotic variability. Note the presence of windows of stability with one attracting limit cycle.

Power spectrum w.r.t. ϵ: deterministic *vs.* stochastic

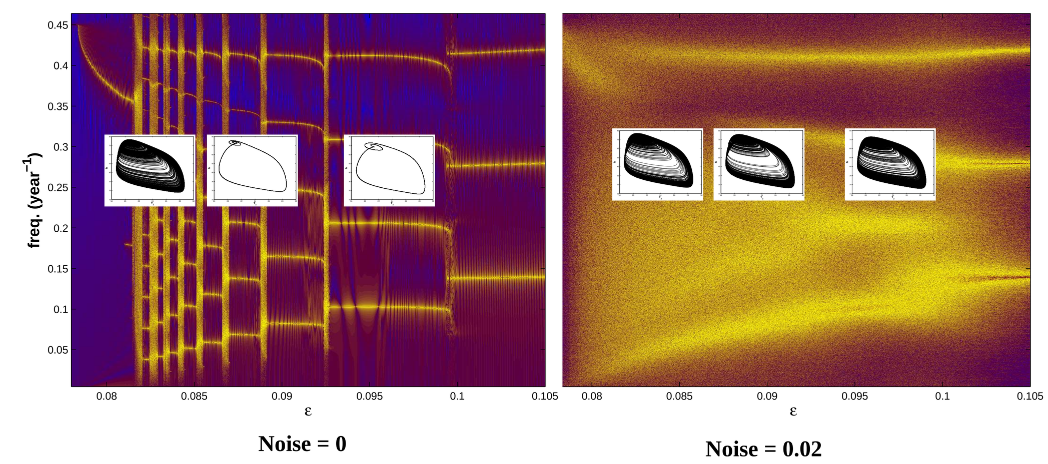


Figure 4: Chaotic variability — interannual-to-interdecadal.

6. Stochastic dynamics: Numerical results

- Below are random attractors for two different noise levels and different values ϵ of zonal advection.
- Golden areas are most frequently visited (log-scale). The measures μ_ω are **nearly singular**.

Random (Shilnikov) horseshoes

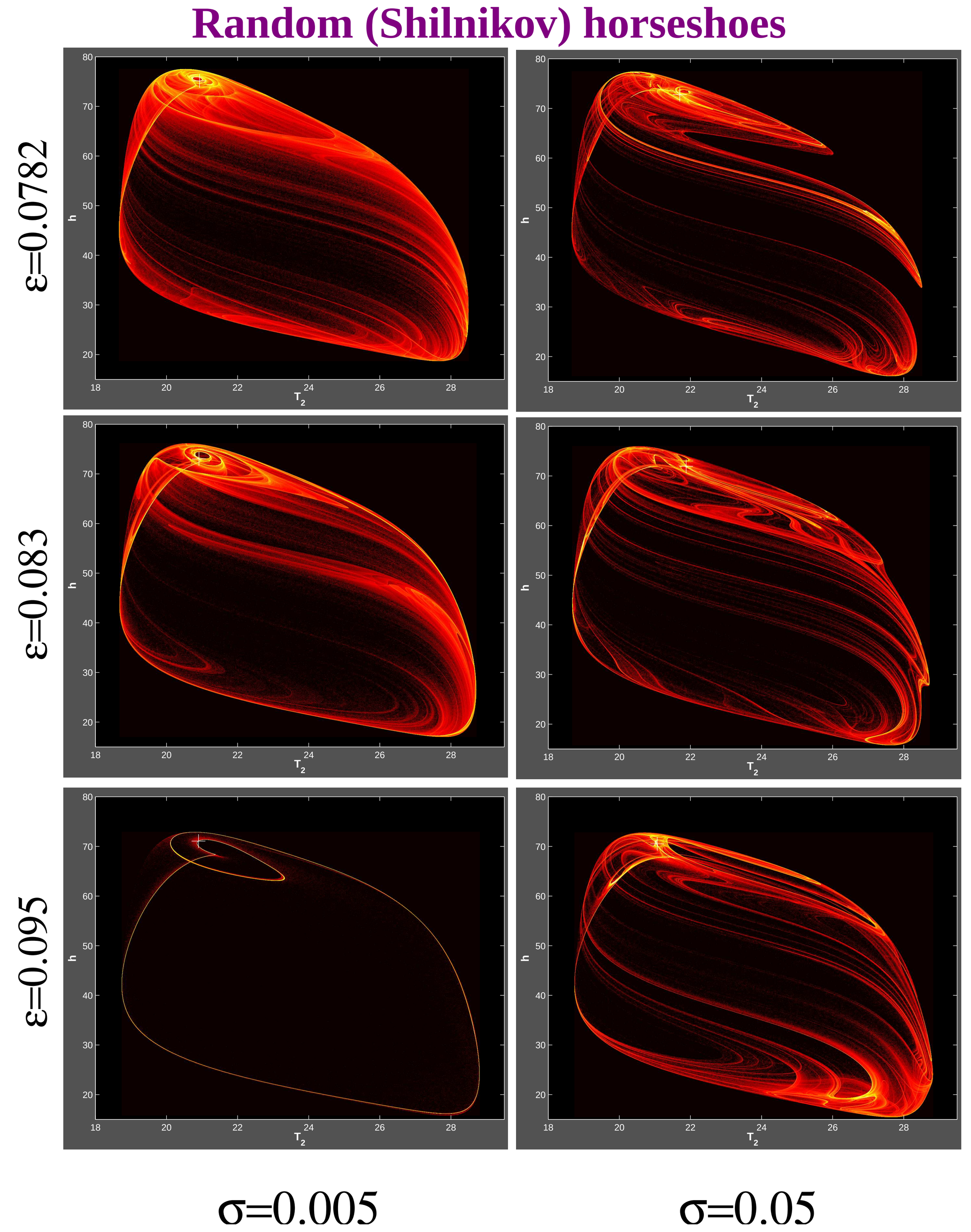


Figure 5: The perturbed deterministic regimes are $\epsilon = 0.0782$, a stable fixed point; $\epsilon = 0.083$, a homoclinic trajectory; and $\epsilon = 0.095$, a twisted limit cycle.

- Horseshoes can be noise-excited even for ϵ-values for which the deterministic dynamics exhibits an attracting fixed point, provided the amount of noise is sufficiently large.

7. Random attractor and predictability

An episode in the random attractor's life for the TJ model

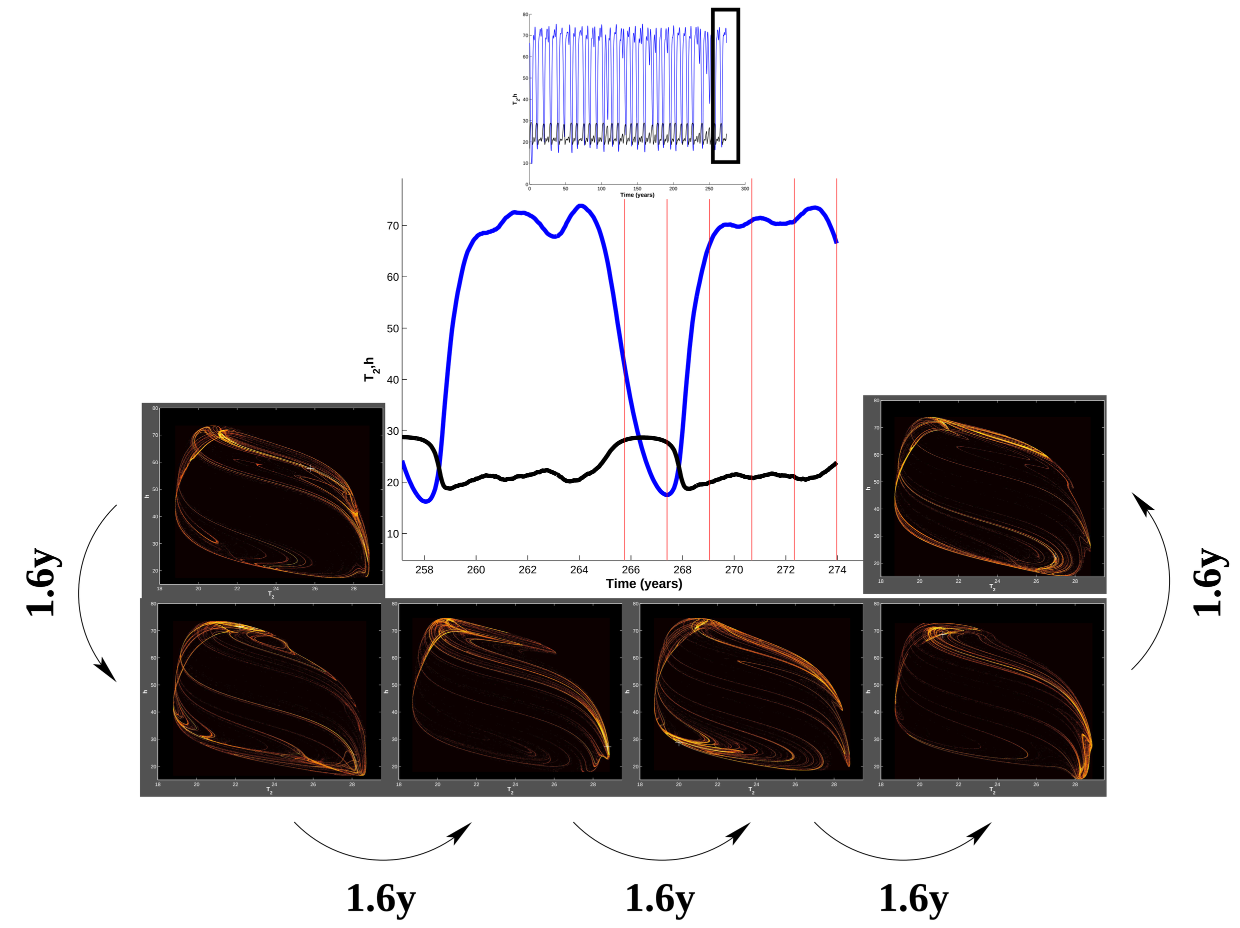


Figure 6: Time series of the thermocline (blue) and the eastern temperature (**black**).
The random attractors are computed at equal intervals, for each red section.

- Statistics at time t given by $\mu_{\theta(t)\omega}$ are evolving with time in an intricate way: they provide crucial information on extremes, and the most probable events!
- **TJ model**: the evolution of μ_ω's maximum (white 'plus') provides a clear prediction of El Niño (high T_2 and low h) and La Niña (low T_2 and high h) episodes, for a given realization.
- In theory, slight changes in initial data result in very different locations at time t, due to *positive Lyapunov exponents*. Random attractor movies indicate that for the **TJ** model, the dependence on the initial data is weak.
- These statistics μ_ω are robust w.r.t. perturbations of the dynamics.

8. Concluding remarks and outlook

- Random attractors give a clear geometric view of the dynamics in the stochastic context.
- For high-dimensional ENSO models, approximate random attractors can be computed. A low-dimensional, physically relevant projection can be derived by using the **available potential energy** E of the tropical Pacific Ocean, and W, the **work** done on that basin by the winds.
- The random attractor and its measure can help design data assimilation procedures. They provide the exact location of the statistics of the stochastic model w.r.t. to a set of initial data in the past and for a given realization ω.

References
- Ghil, M., M.D. Chekroun, and E. Simonnet, 2008: Climate dynamics and fluid mechanics: Natural variability and related uncertainties, *Physica D*, **237**, 2111–2126.
- Timmermann, A., and F-F. Jin, A nonlinear mechanism for decadal El Niño amplitude changes, *Geophys. Res. Lett.*, **29** (1), 2002.