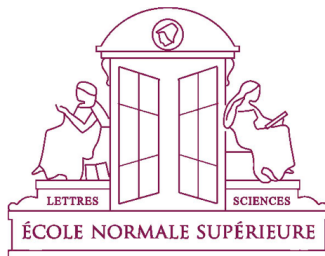


Toward a Mathematical Theory of Climate Sensitivity

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**Joint work with M.D. Chekroun and D. Kondrashov (UCLA),
E. Simonnet (INLN, Nice), S. Wang (Indiana U.)
and I. Zaliapin (U. Nevada, Reno)**



Please visit these sites for more info.

<http://www.atmos.ucla.edu/tcd/>

<http://www.environnement.ens.fr/>

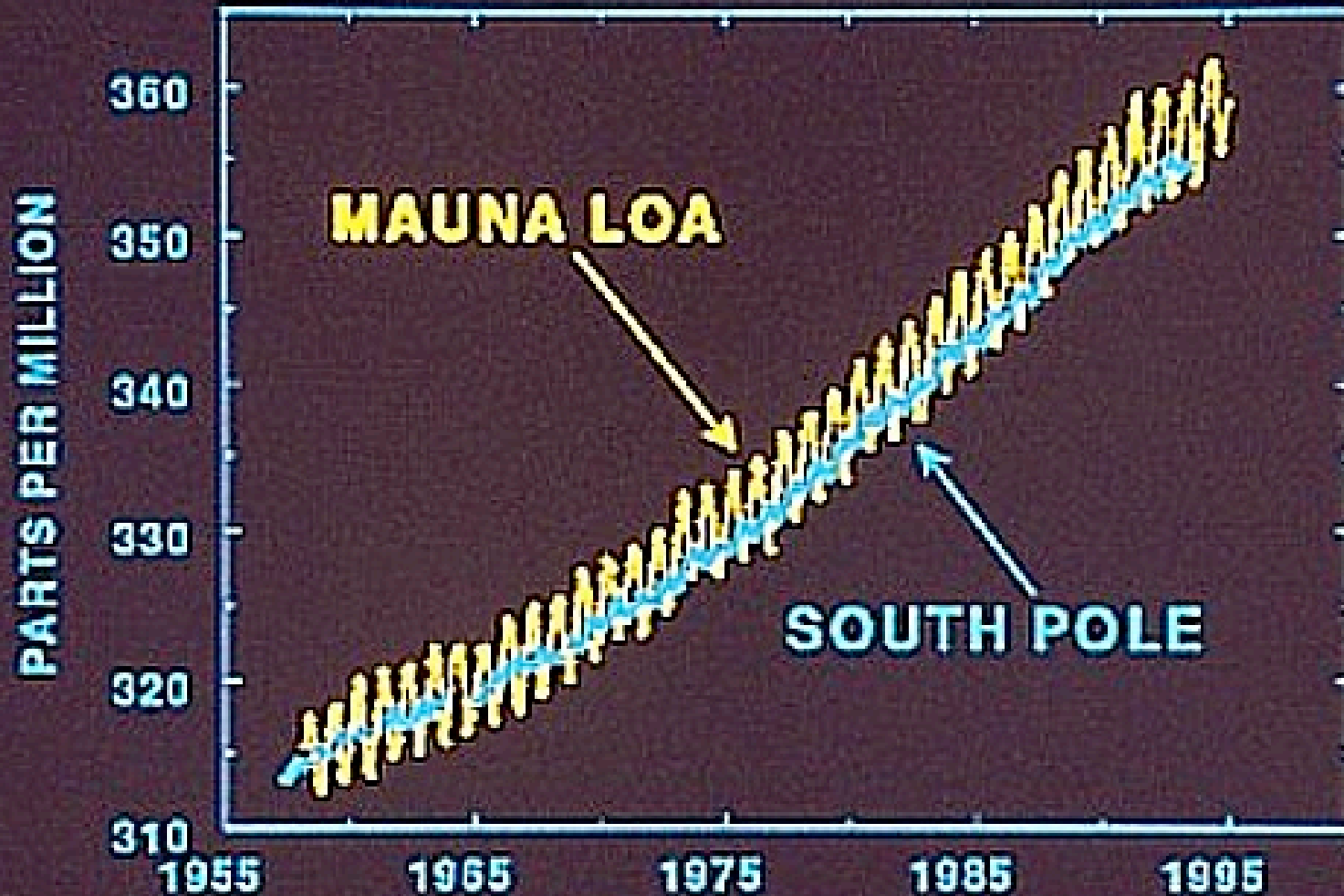
Motivation

- The **climate system** is highly **nonlinear and** quite **complex**.
- Its **major components** – the atmosphere, oceans, ice sheets – **evolve on many** space and time **scales**.
- Its **predictive understanding** has to rely on the system's physical, chemical and biological **modeling**, but also on the **mathematical analysis** of the models thus obtained.
- The **hierarchical modeling** approach allows one to give proper weight to the **understanding** provided by the models **vs.** their **realism**, respectively.
- Back-and-forth between **“toy”** (conceptual) and **detailed** (“realistic”) **models**, and between **models** and **data**.
- Such an approach facilitates the evaluation of **forecasts** (**prognostications?**) based on these models.

Outline

- The IPCC process: results and further questions
- Natural climate variability as a source of uncertainties
 - sensitivity to initial data → error growth
 - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
 - structural in/stability
 - random dynamical systems (RDS)
- Two or more illustrative examples
 - Arnol'd tongues and a “French garden”
 - the Lorenz system
 - an ENSO “toy” model
- Linear response theory and climate sensitivity
- Conclusions, work in progress and references

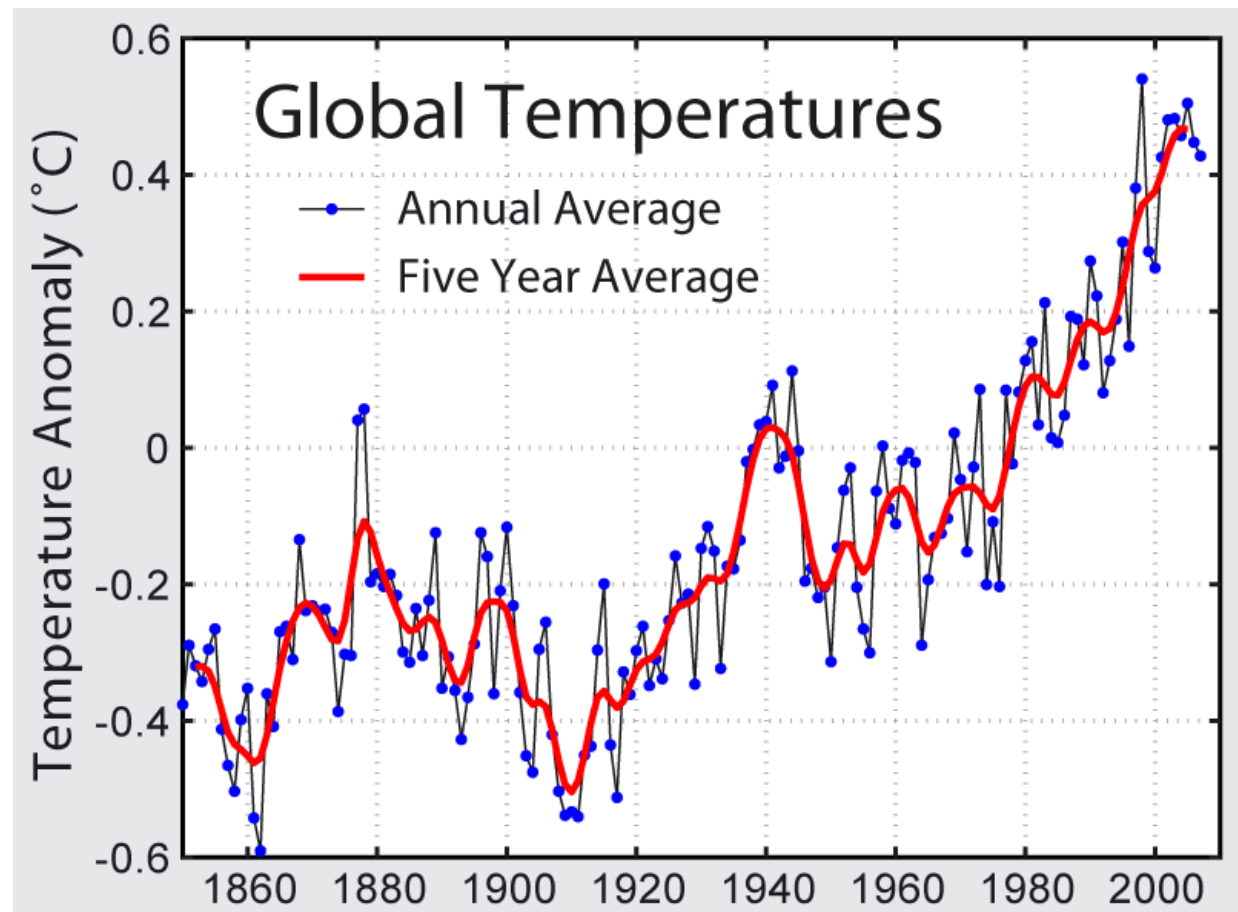
CO2 IN THE ATMOSPHERE



Temperatures and GHGs

Greenhouse gases (GHGs) go up,
temperatures go up:

It's gotta do with us, at least a bit,
doesn't it?



Wikicommons, from
Hansen *et al.* (*PNAS*, 2006);
see also <http://data.giss.nasa.gov/gistemp/graphs/>

Unfortunately, things aren't all that easy!

What to do?

Try to achieve better interpretation of, and agreement between, models ...

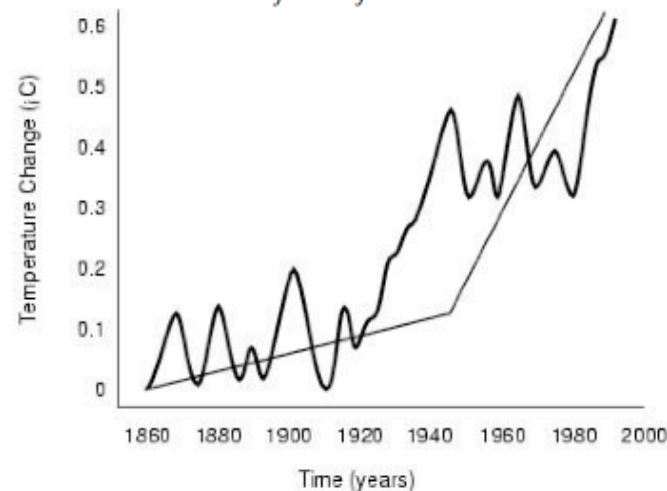
Ghil, M., 2002: Natural climate variability, in *Encyclopedia of Global Environmental Change*, T. Munn (Ed.), Vol. 1, Wiley

Natural variability introduces additional complexity into the anthropogenic climate change problem

The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

$$c \frac{dT}{dt} = -kT + Q$$

$k = \sum k_i$ – feedbacks (+ve and -ve)
 $Q = \sum Q_j$ – sources & sinks
 $Q_j = Q_j(t)$



Linear response to CO₂ vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

$$\frac{dX}{dt} = N(X, t, \mu, \beta)$$

Global warming and its socio-economic impacts

Temperatures rise:

- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, *i.e.*, it depends on the accuracy and reliability of the forecast ...

Source : IPCC (2007),
AR4, WGI, SPM

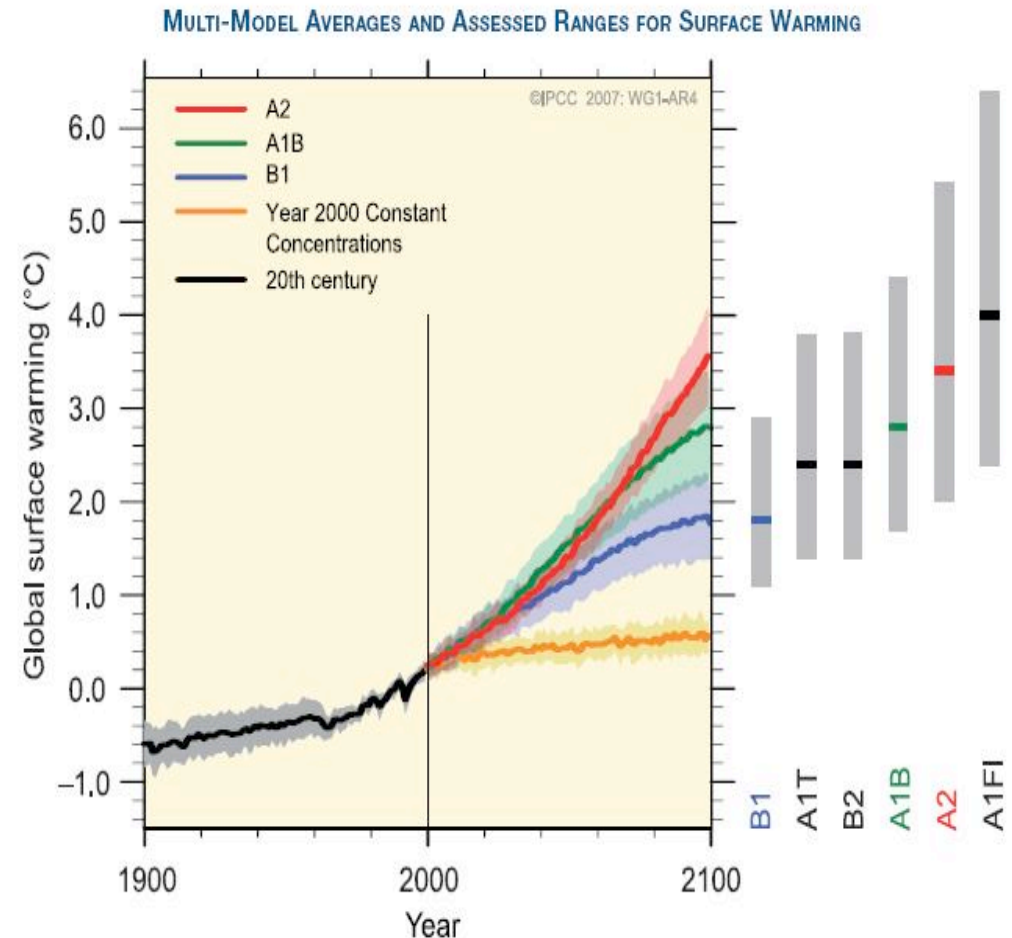
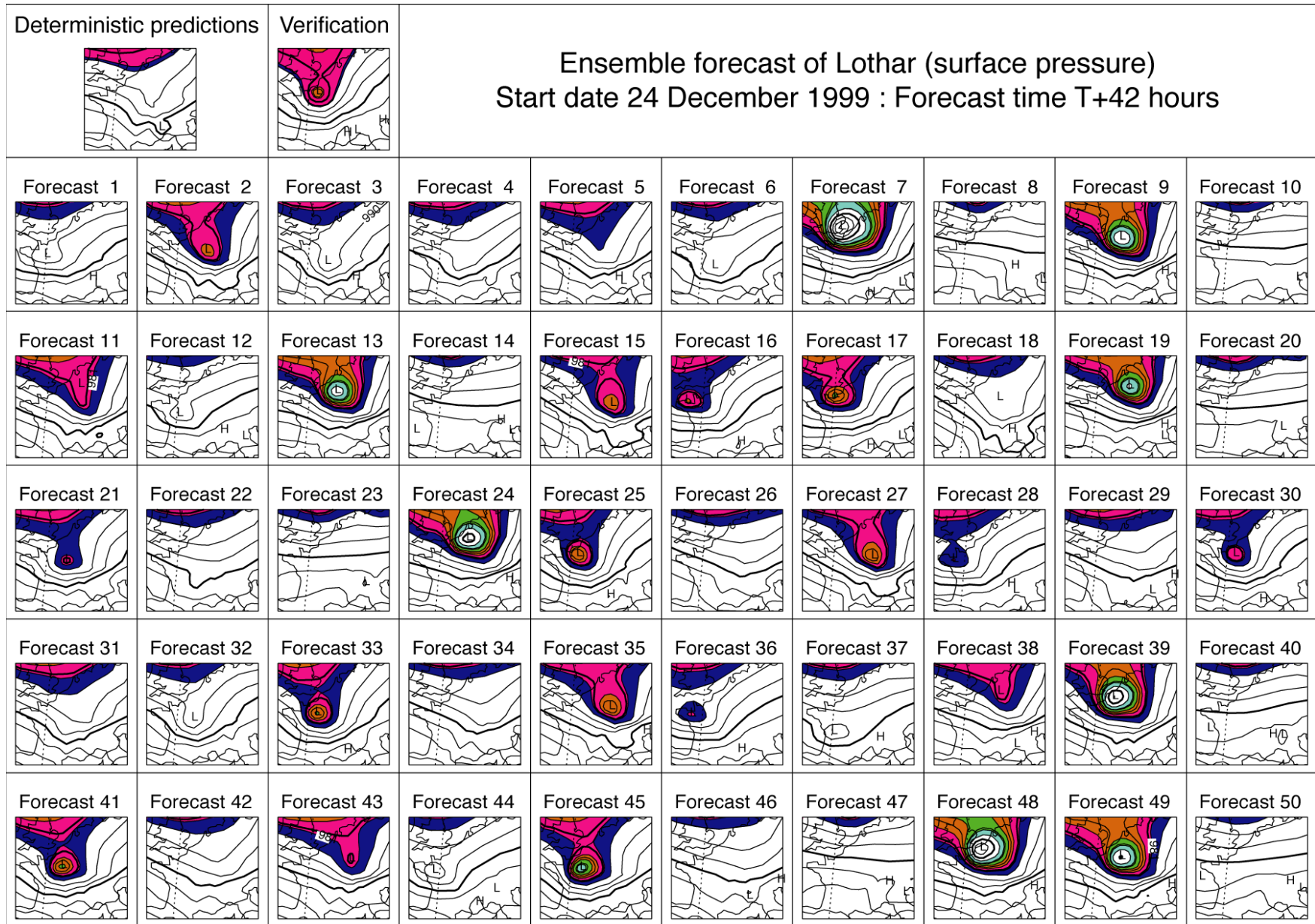


Figure SPM.5. Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ± 1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the likely range assessed for the six SRES marker scenarios. The assessment of the best estimate and likely ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. (Figures 10.4 and 10.29)

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Courtesy Tim Palmer, 2009

So what's it gonna be like, by 2100?

Table SPM.2. Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend. (Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9)

Phenomenon ^a and direction of trend	Likelihood that trend occurred in late 20th century (typically post 1960)	Likelihood of a human contribution to observed trend ^b	Likelihood of future trends based on projections for 21st century using SRES scenarios
Warmer and fewer cold days and nights over most land areas	<i>Very likely^c</i>	<i>Likely^d</i>	<i>Virtually certain^d</i>
Warmer and more frequent hot days and nights over most land areas	<i>Very likely^e</i>	<i>Likely (nights)^d</i>	<i>Virtually certain^d</i>
Warm spells/heat waves. Frequency increases over most land areas	<i>Likely</i>	<i>More likely than not^f</i>	<i>Very likely</i>
Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas	<i>Likely</i>	<i>More likely than not^f</i>	<i>Very likely</i>
Area affected by droughts increases	<i>Likely in many regions since 1970s</i>	<i>More likely than not</i>	<i>Likely</i>
Intense tropical cyclone activity increases	<i>Likely in some regions since 1970</i>	<i>More likely than not^f</i>	<i>Likely</i>
Increased incidence of extreme high sea level (excludes tsunamis) ^g	<i>Likely</i>	<i>More likely than not^h</i>	<i>Likelyⁱ</i>

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Can we, nonlinear dynamicists, help?

The uncertainties
might be *intrinsic*,
rather than mere
“tuning problems”

If so, maybe
*stochastic structural
stability* could help!

Might fit in nicely with
recent taste for
“stochastic
parameterizations”

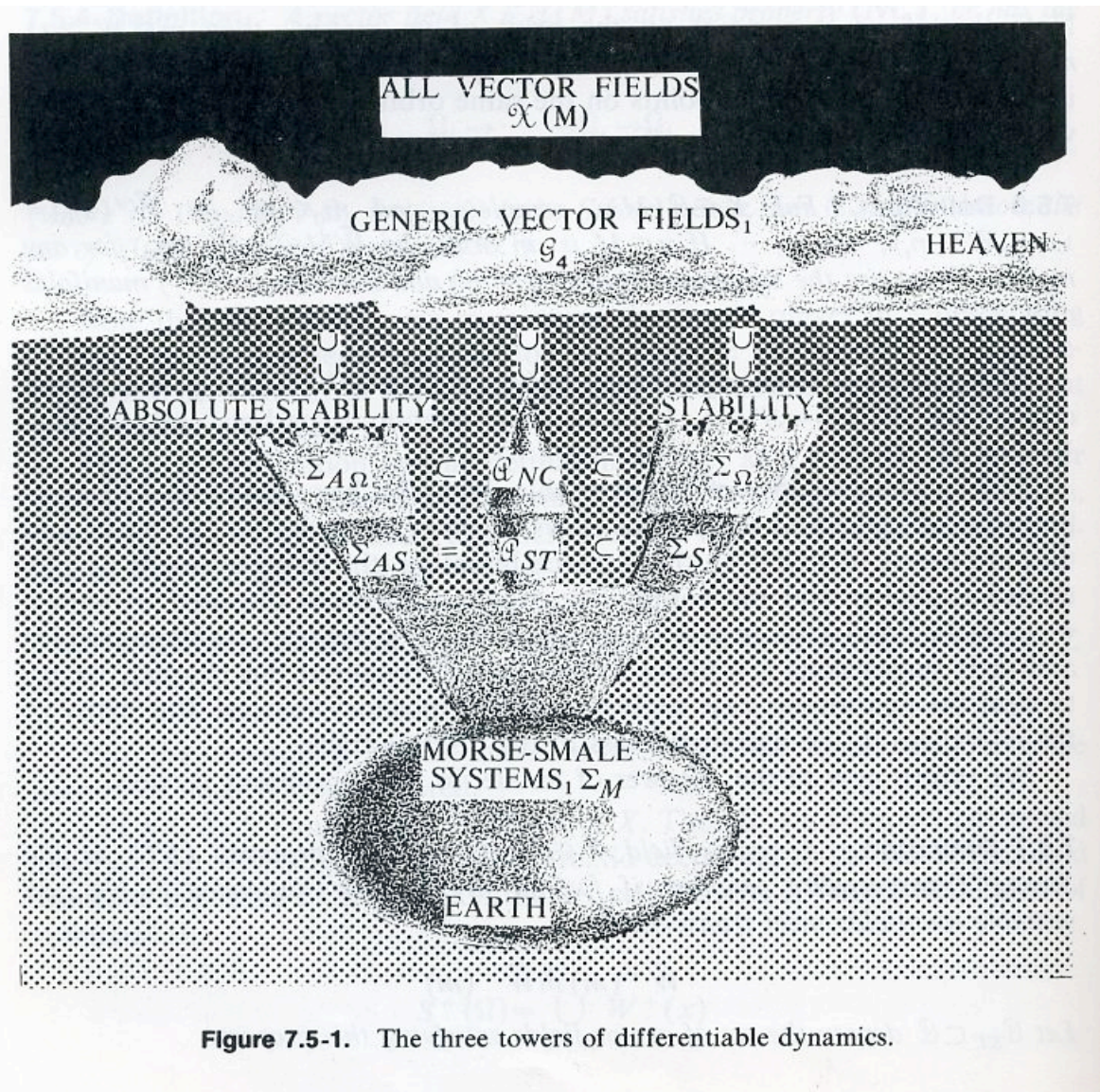
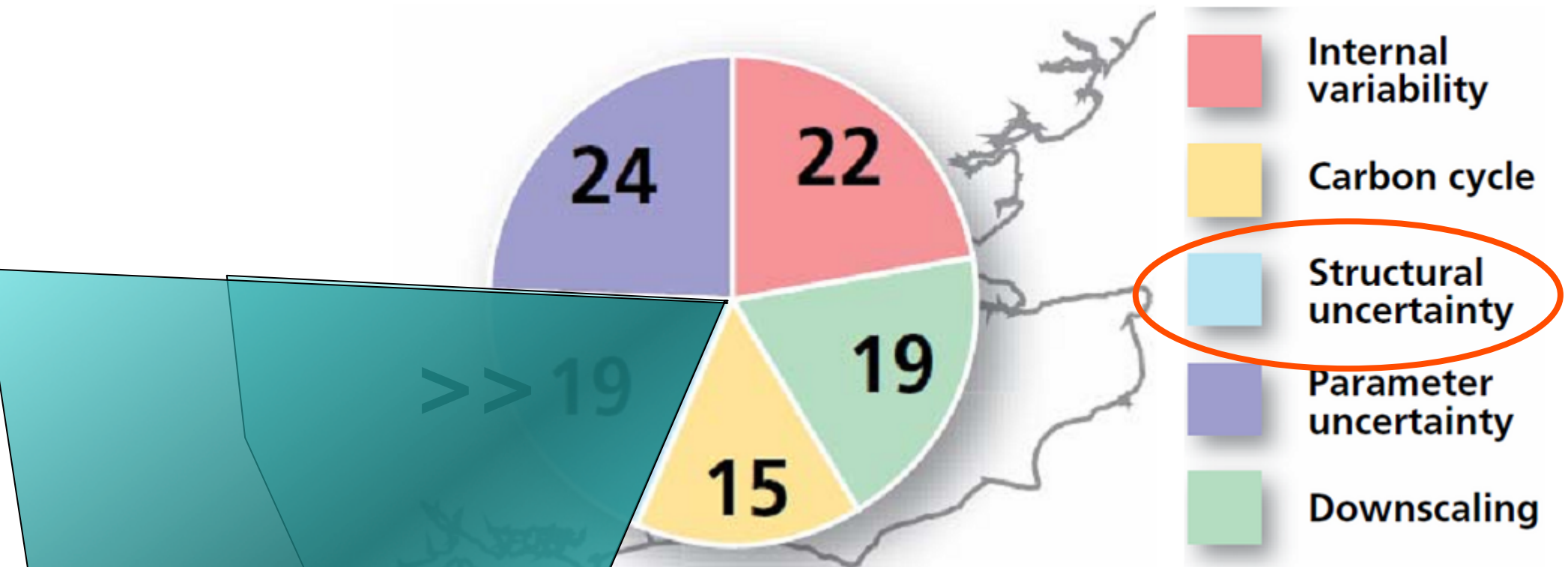


Figure 7.5-1. The three towers of differentiable dynamics.

The DDS dream of structural stability (from Abraham & Marsden, 1978)

How important are different sources of uncertainty?

- Varies, but typically no single source dominates.



Uncertainties in winter precipitation changes for the 2080s relative to 1961-90, at a 25km box in SE England

Source: Met Office

Non-autonomous Dynamical Systems - I

A linear example as a paradigm

Let us first start with a **very difficult problem**:

$$\text{Study the "dynamics" of } \dot{x} = -\alpha x + \sigma t, \quad \alpha, \sigma > 0. \quad (1)$$

First remarks:

- The system $\dot{x} = -\alpha x$, i.e. the autonomous part of (1), is **dissipative**. All the solutions of $\dot{x} = -\alpha x$ converge to 0 as $t \rightarrow +\infty$.
- Is it the case for (1)? Certainly not!
The **autonomous part** is **forced**; we even introduce an **infinite energy** over an infinite time interval: $\int_0^{+\infty} t \, dt = +\infty!$
Forward attraction seems to be ill adapted to time-dependent forcing.

Goal:

Find a concept of attraction that is:

- (i) compatible with the forward concept, when there is no forcing; and
- (ii) provides a way to assess the effect of dissipation in some sense.

For that let's do some computations...

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For that let's do some computations...

Remarks

- We've just shown that:

$$|x(t, s; x_0) - a(t)| \xrightarrow{s \rightarrow -\infty} 0 ; \text{ for every } t \text{ fixed,}$$

and for all initial data x_0 , with $a(t) = \frac{\sigma}{\alpha}(t - 1/\alpha)$.

- We've just encountered the concept of **pullback attraction**; here $\{a(t)\}$ is the **pullback attractor** of the system (1).
- What does it mean physically?

The pullback attractor provides a way to assess an **asymptotic regime at time t** — the time at which we observe the system — for a system starting to evolve from the remote past s , $s \ll t$.

- This asymptotic regime evolves with time: it is a dynamical object.
- Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.

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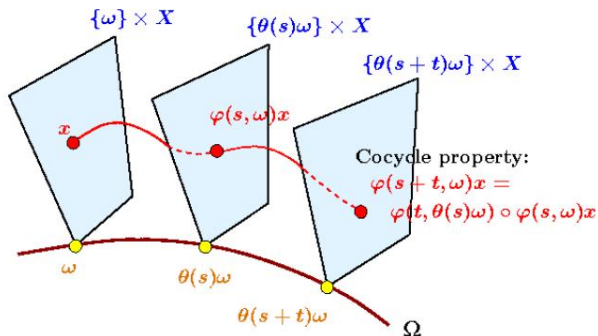
Random Dynamical Systems - RDS theory

This theory is a combination of measure (probability) theory and dynamical systems, treated systematically by the “Bremen group” (L.Arnold, 1998). It allows one to treat stochastic differential equations (**SDEs**), and more general systems driven by some “noise,” as **flows**.

Setting:

- (i) A phase space X . **Example:** \mathbb{R}^n .
- (ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example:** The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P} = \gamma$.
- (iii) A model of the noise $\theta(t) : \Omega \rightarrow \Omega$ that preserves the measure \mathbb{P} , i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; θ is called **the driving system**. **Example:** $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$; it starts the noise at s instead of $t = 0$.
- (iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \rightarrow X$ with the cocycle property. **Example:** The solution of an SDE.

Random Dynamical Systems - A geometric view of SDEs



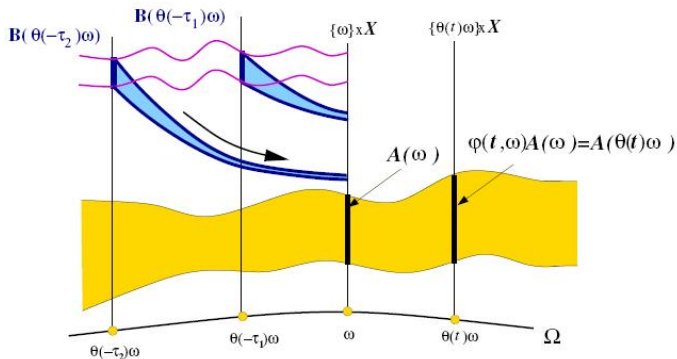
- φ is a random dynamical system (RDS)
- $\Theta(t)(x, \omega) = (\theta(t)\omega, \varphi(t, \omega)x)$ is a flow on the bundle

Random Dynamical Systems - Random attractor

A random attractor $\mathcal{A}(\omega)$ is both *invariant* and “pullback” attracting:

- (a) **Invariant:** $\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$.
- (b) **Attracting:** $\forall B \subset X, \lim_{t \rightarrow \infty} \text{dist}(\varphi(t, \theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$ a.s.

Pullback attraction to $\mathcal{A}(\omega)$



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Short description of the deterministic model

- Dynamics on a 2-D torus:

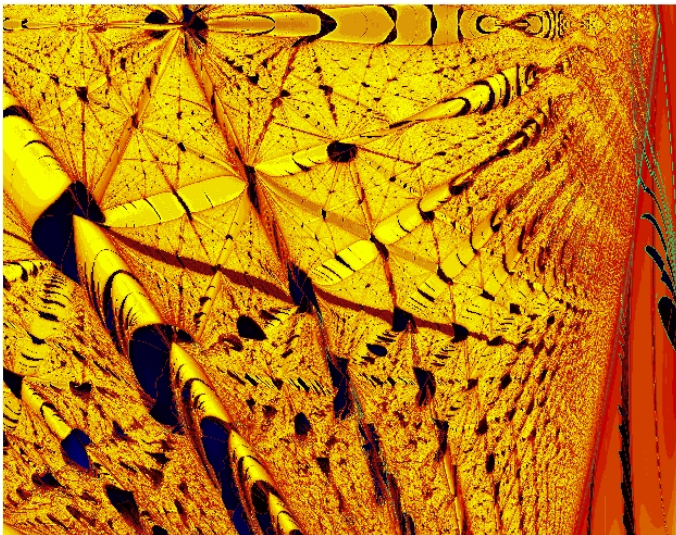
$$\begin{aligned}x_{n+1} &= x_n + \Omega_1 - \varepsilon \sin(2\pi y_n), & \text{mod } 1 \\y_{n+1} &= y_n + \Omega_2 - \varepsilon \sin(2\pi x_n) & \text{mod } 1\end{aligned}$$

- **Web of resonances & chaos:**
 - **Partial resonance** (Ω_1, Ω_2 are rational and there is one rational relation $m_1\Omega_1 + m_2\Omega_2 = k \in \mathbb{Z}^*$ with $(m_1, m_2) \in \mathbb{Z}^* \times \mathbb{Z}^*$)
 - **Full resonance**
 - **Chaos** with possibly multiple attractors
- A more realistic paradigm of observed dynamics in the geosciences, and more...
- What is the effect of noise in such a context?

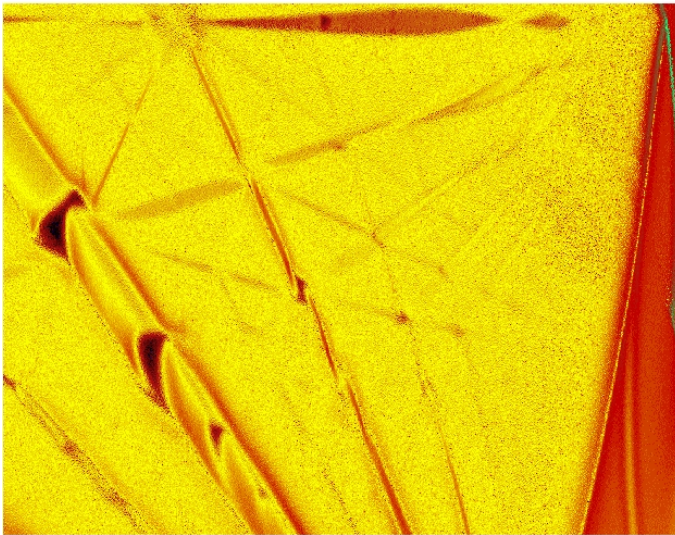
A French garden near the castle of La Roche-Guyon



Devil's quarry for a coupling parameter $\varepsilon = 0.15$: a web of resonances

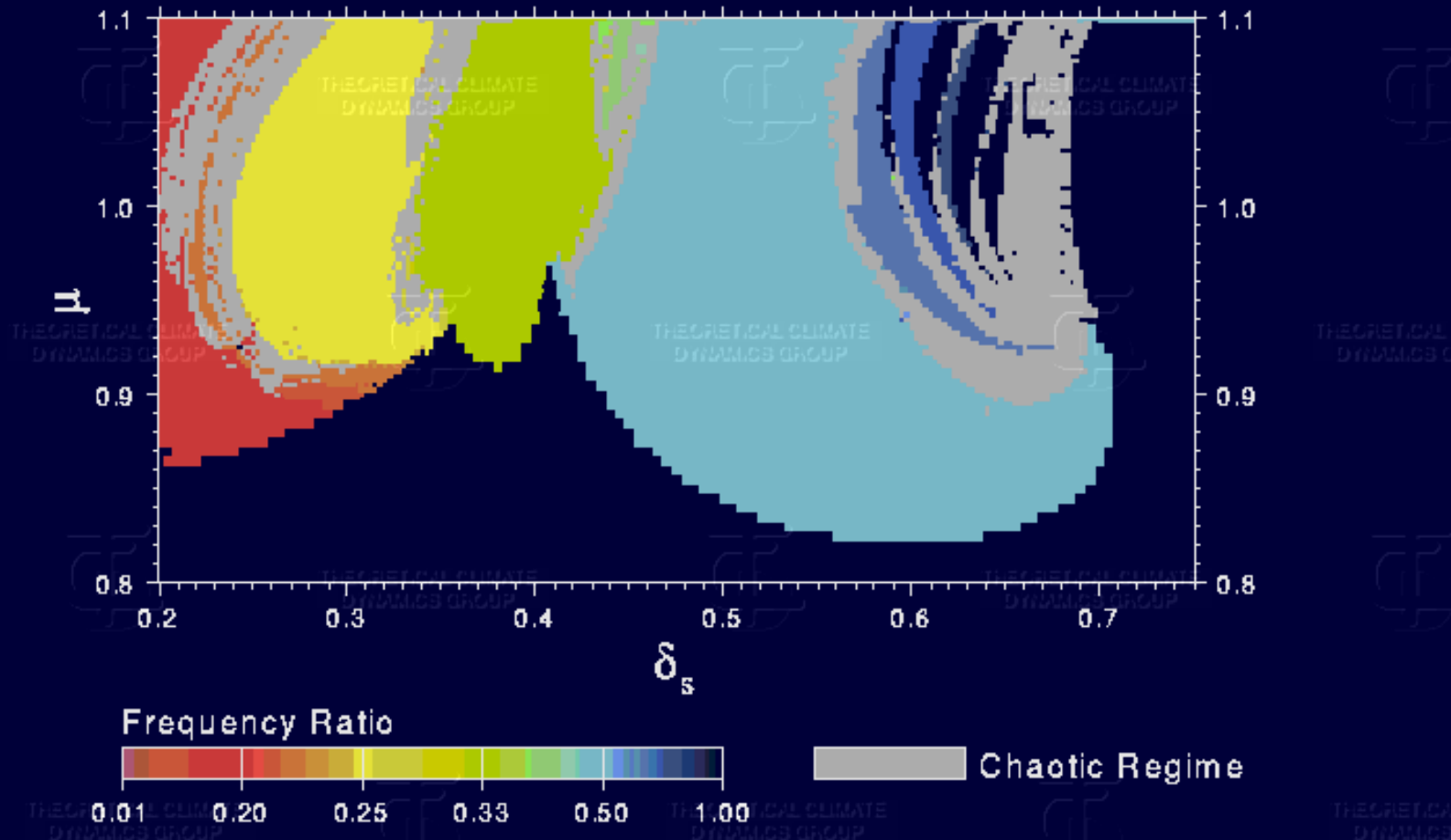


Effect of the noise on Devil's quarry



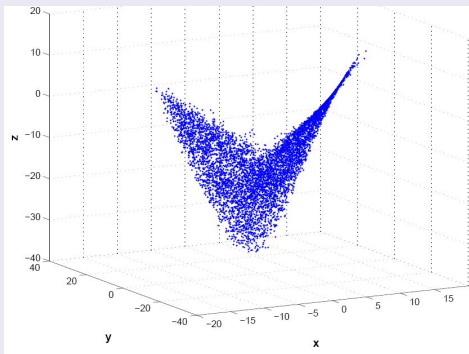
Devil's Bleachers in a 1-D ENSO Model

Ratio of ENSO frequency to annual cycle



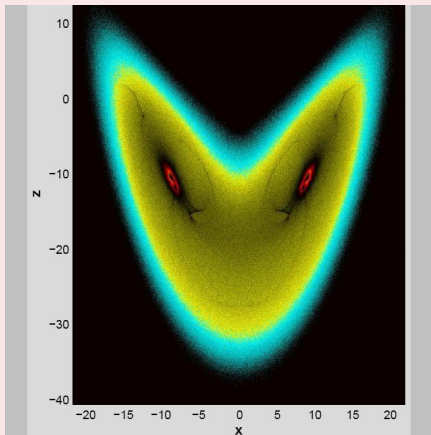
Random attractor of the stochastic Lorenz system

Snapshot of the random attractor (RA)



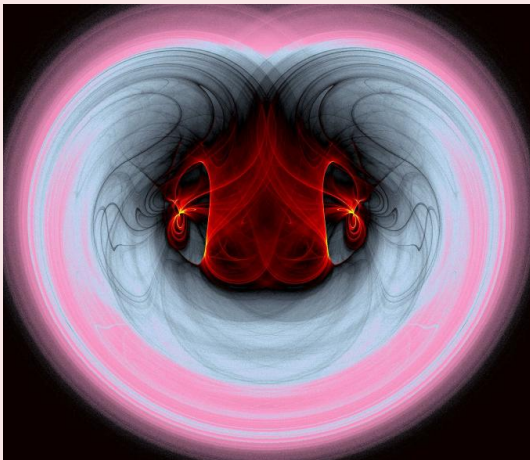
- A **snapshot** of the RA, $\mathcal{A}(\omega)$, computed at a fixed time t and for the **same realization** ω ; it is made up of points transported by the stochastic flow, from the remote past $t - T$, $T \gg 1$.
- We use **multiplicative noise** in the deterministic Lorenz model, with the classical parameter values $b = 8/3$, $\sigma = 10$, and $r = 28$.
- Even computed **pathwise**, this object supports meaningful **statistics**.

Sample measures supported by the R.A.



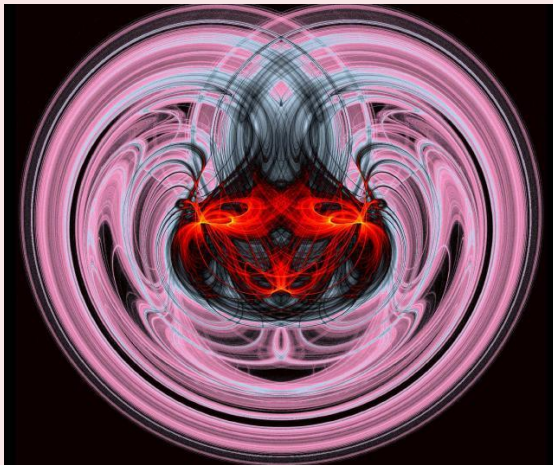
- We compute the probability measure on the R.A. at some fixed time t , and for a fixed realization ω . We show a “projection”, $\int \mu_\omega(x, y, z) dy$, with **multiplicative noise**: $dx_i = \text{Lorenz}(x_1, x_2, x_3) dt + \alpha x_i dW_t; i \in \{1, 2, 3\}$.
- **10 million of initial points** have been used for this picture!

Sample measure supported by the R.A.



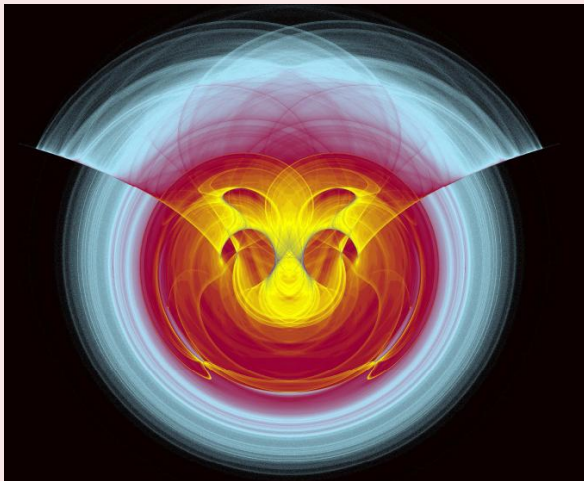
- Still 1 Billion I.D., and $\alpha = 0.3$.

Sample measure supported by the R.A.



- Still 1 Billion I.D., and $\alpha = 0.5$. Another one?

Sample measure supported by the R.A.



- Here $\alpha = 0.4$. The sample measure is approximated for another realization ω of the noise, starting from 8 billion I.D.
- Now more serious stuff is coming...

Sample measures evolve with time.

- Recall that these sample measures are the **frozen statistics** at a time t for a realization ω .
- How do these **frozen statistics** evolve with time?
- **Action!**

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Applications to a nonlinear stochastic El Niño model

Chekroun, Simonnet and Ghil, 2011

Timmerman & Jin (*Geophys. Res. Lett.*, 2002) have derived the following low-order, tropical-atmosphere–ocean model. The model has three variables: thermocline depth anomaly h , and SSTs T_1 and T_2 in the western and eastern basin.

$$\begin{aligned}\dot{T}_1 &= -\alpha(T_1 - T_r) - \frac{2\varepsilon u}{L}(T_2 - T_1), \\ \dot{T}_2 &= -\alpha(T_2 - T_r) - \frac{w}{H_m}(T_2 - T_{sub}), \\ \dot{h} &= r(-h - bL\tau/2).\end{aligned}$$

The related diagnostic equations are:

$$\begin{aligned}T_{sub} &= T_r - \frac{T_r - T_{r0}}{2} [1 - \tanh(H + h_2 - z_0)/h^*] \\ \tau &= \frac{a}{\beta} (T_1 - T_2) [\xi_t - 1].\end{aligned}$$

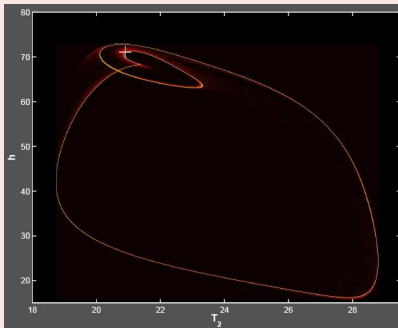
- τ : the wind stress anomalies, $w = -\beta\tau/H_m$: the equatorial upwelling.
- $u = \beta L\tau/2$: the zonal advection, T_{sub} : the subsurface temperature.

Wind stress bursts are modeled as white noise ξ_t of variance σ , and ε measures the strength of the **zonal advection**.

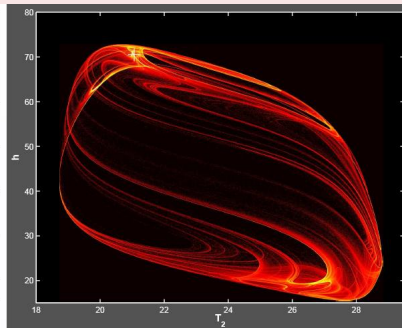


Random attractors: the frozen statistics

Random Shil'nikov horseshoes



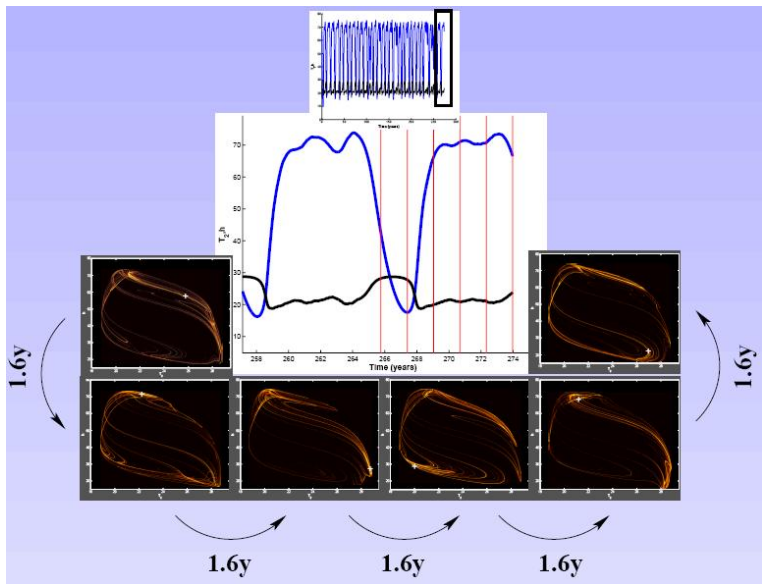
$$\sigma=0.005$$



$$\sigma=0.05$$

- Horseshoes can be noise-excited, left: a weakly-perturbed limit cycle, right: the same with larger noise.
- Golden: most frequently-visited areas; white 'plus' sign: most visited.

An episode in the random's attractor life



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Climate and Its Sensitivity

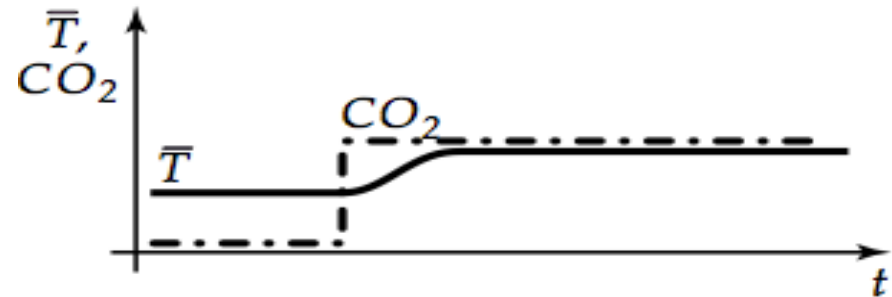
Let's say CO₂ doubles:

How will “climate” change?

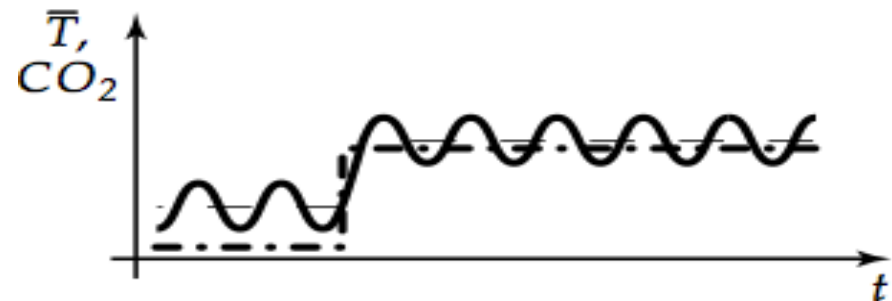
1. Climate is in **stable equilibrium** (fixed point); if so, **mean temperature** will just shift gradually to its new equilibrium value.
2. Climate is **purely periodic**; if so, **mean temperature** will (maybe) shift gradually to its new equilibrium value. But how will the **period, amplitude and phase** of the **limit cycle** change?
3. And how about some “real stuff” now: **chaotic + random**?

Ghil (Encycl. Global Environmental Change, 2002)

a) *Equilibrium sensitivity*



b) *Nonequilibrium sensitivity*



The Ruelle response formula

- Physically, the challenge is to find the trade-off between the physics present in the model and the stochastic parameterizations of the missing physics.
From a mathematical point of view, climate sensitivity can be related to **sensitivity of SRB measures**.
- The **thermodynamic formalism à la Ruelle, in the RDS context**, helps to understand the response of **systems out-of-equilibrium**, to changes in the parameterizations (Gundlach, Kifer, Liu).
- **The Ruelle response formula**: Given an SRB measure μ of an autonomous chaotic system $\dot{x} = f(x)$, an **observable** $G : X \rightarrow \mathbb{R}$, and a smooth time-dependent perturbation X_t , the time-dependent variations $\delta_t \mu$ of μ are given by:

$$\delta_t \mu(G) = \int_{-\infty}^t d\tau \int \mu(dx) X_\tau(x) \cdot \nabla_x (G \circ \varphi_{t-\tau}(x)),$$

where φ_t is the flow of the unperturbed system $\dot{x} = f(x)$.

The susceptibility function

- In the case $X_t(x) = \phi(t)X(x)$, the Ruelle response formula can be written:

$$\delta_t \mu(G) = \int dt' \kappa(t - t') \phi(t'),$$

where κ is called the **response function**. The **Fourier transform** $\hat{\kappa}$ of the response function is called the **susceptibility function**.

- In this case $\delta_t \mu(\hat{G})(\xi) = \hat{\kappa}(\xi) \hat{\phi}(\xi)$ and since the r.h.s. is a product, there are no frequencies in the linear response that are not present in the signal.
- In general, the situation can be more complicated and the theory gives the following criterion of high sensitivity:
℄: Poles of the susceptibility function $\hat{\kappa}(\xi)$ in the upper-half plane
⇒ High sensitivity of the system's response function $\kappa(t)$.
- RDS theory offers a path for extending this criterion when random perturbations are considered.

Sample measures for an NDDE model of ENSO

The Galanti-Tziperman (GT) model (JAS, 1999)

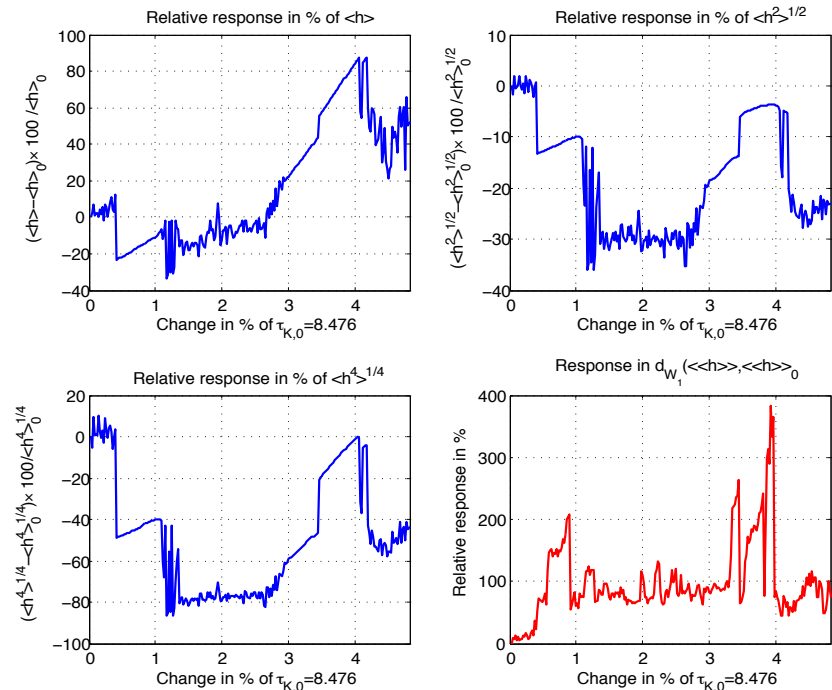
$$\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),$$

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: T is East-basin SST and h is thermocline depth.

$$h(t) = M_1 e^{-\epsilon_m(\tau_1 + \tau_2)} h(t - \tau_1 - \tau_2) - M_2 \tau_1 e^{-\epsilon_m(\frac{\tau_1}{2} + \tau_2)} \mu(t - \tau_2 - \frac{\tau_1}{2}) T(t - \tau_2 - \frac{\tau_1}{2}) + M_3 \tau_2 e^{-\epsilon_m \frac{\tau_2}{2}} \mu(t - \frac{\tau_2}{2}) T(t - \frac{\tau_2}{2}).$$

Seasonal forcing given by $\mu(t) = 1 + \epsilon \cos(\omega t + \phi)$.
The pullback attractor and its invariant measures were computed.

Figure shows the changes in the mean, 2nd & 4th moment of $h(t)$, along with the Wasserstein distance d_W , for changes of 0–5% in the delay parameter $\tau_{K,0}$.



Note intervals of both **smooth** & **rough** dependence!

Pullback attractor and invariant measure of the GT model

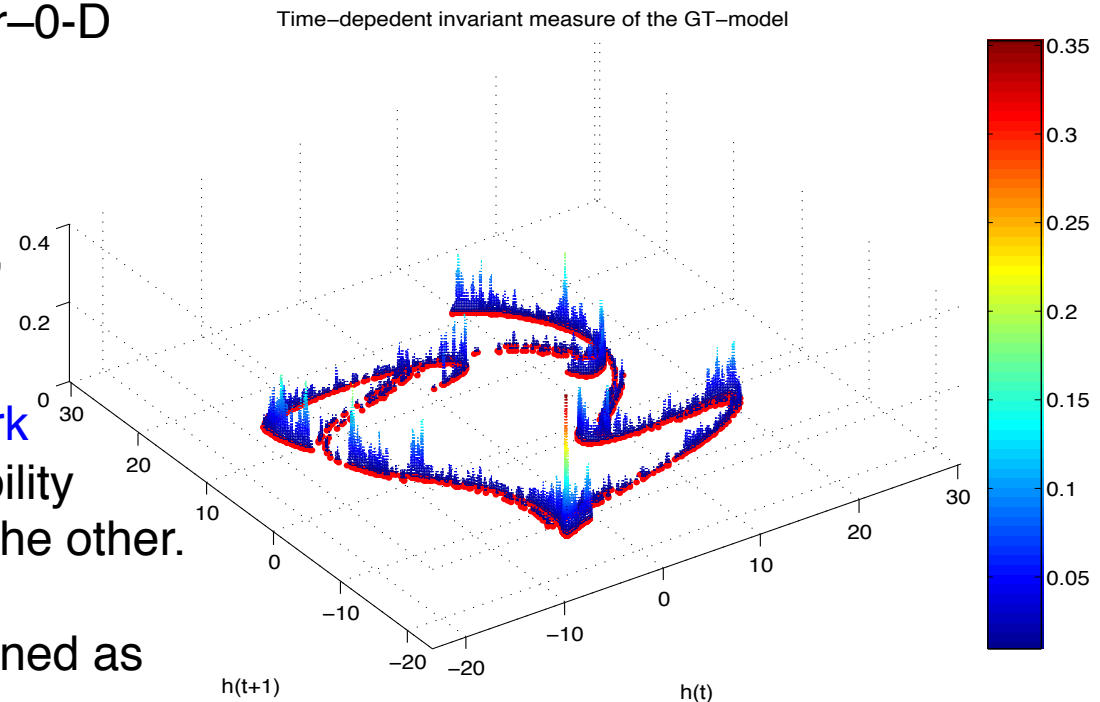
The **time-dependent pullback attractor** of the GT model supports a time-dependent **invariant measure**, whose density is plotted in 3-D perspective.

The plot is in delay coordinates $h(t+1)$ vs. $h(t)$ and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near-0-D peaks on these filaments.

The **Wasserstein distance** d_W between one such configuration, at given parameter values, and another one, at a different set of values, **is proportional to the work** needed to move the total probability mass from one configuration to the other.

Climate sensitivity γ can be defined as

$$\gamma = \partial d_W / \partial \tau$$



Outline

- The IPCC process: results and further questions.
- Natural climate variability as a source of uncertainties
 - sensitivity to initial data → error growth
 - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
 - structural in/stability
 - random dynamical systems (RDS)
- Two or more illustrative examples
 - Arnol'd tongues and a “French garden”
 - the Lorenz system
 - an ENSO “toy” model
- Linear response theory and climate sensitivity
- **Conclusions, work in progress and references**

Concluding remarks, I – RDS and RAs

Summary

- A change of paradigm for open, non-autonomous systems
- Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

Work in progress

- Study the effect of specific **stochastic parametrizations** on model robustness.
- Applications to **intermediate models and GCMs**.
- Implications for **climate sensitivity**.
- Implications for **predictability?**

Concluding remarks, II – General

What do we know?

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

What do we know less well?

- By how much?
 - Is it getting warmer ...
 - Do we contribute to it ...
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

What to do?

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
 - stochastic structural and statistical stability!
 - linear response = response function + susceptibility function

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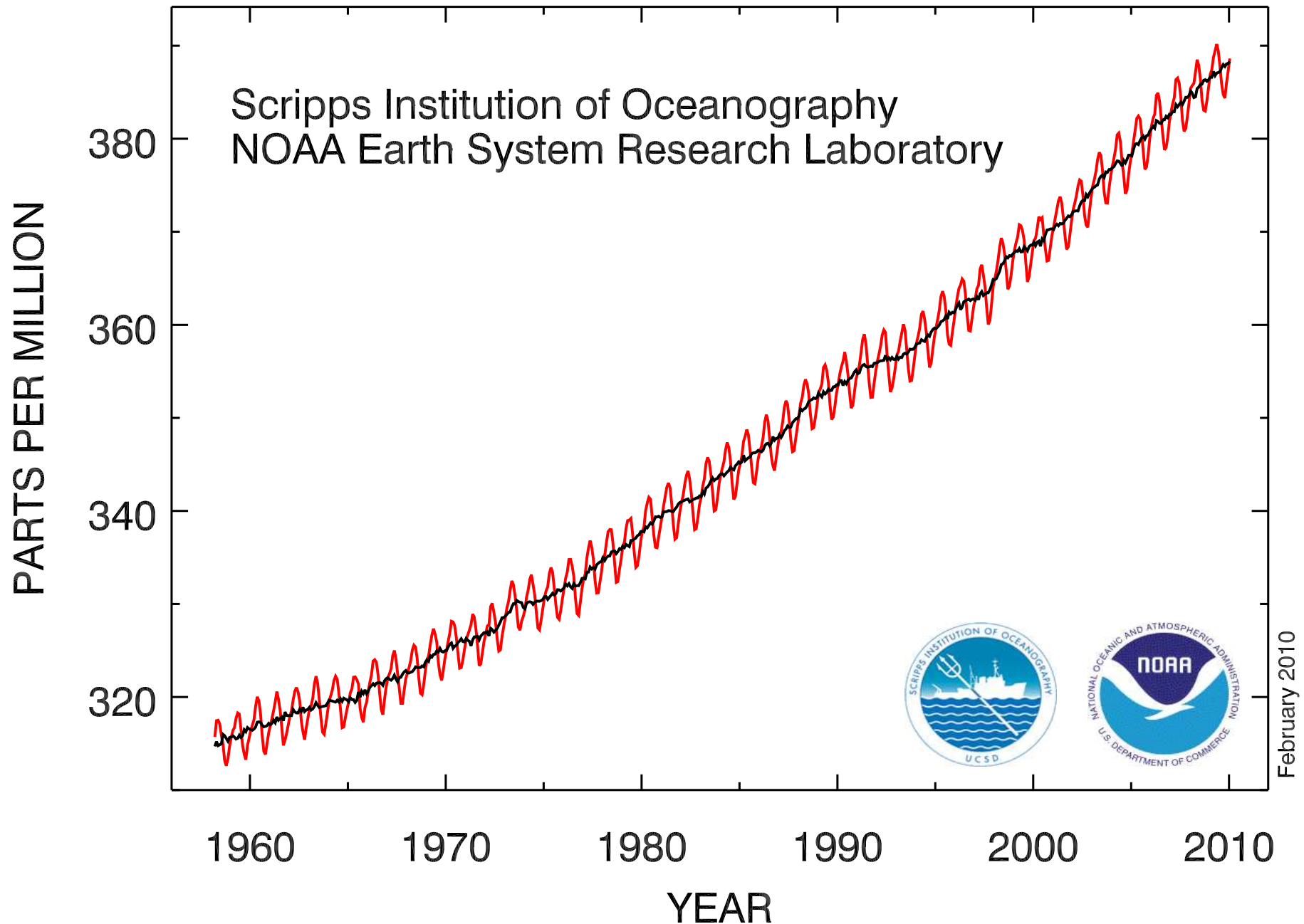
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Some general references

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Reserve slides

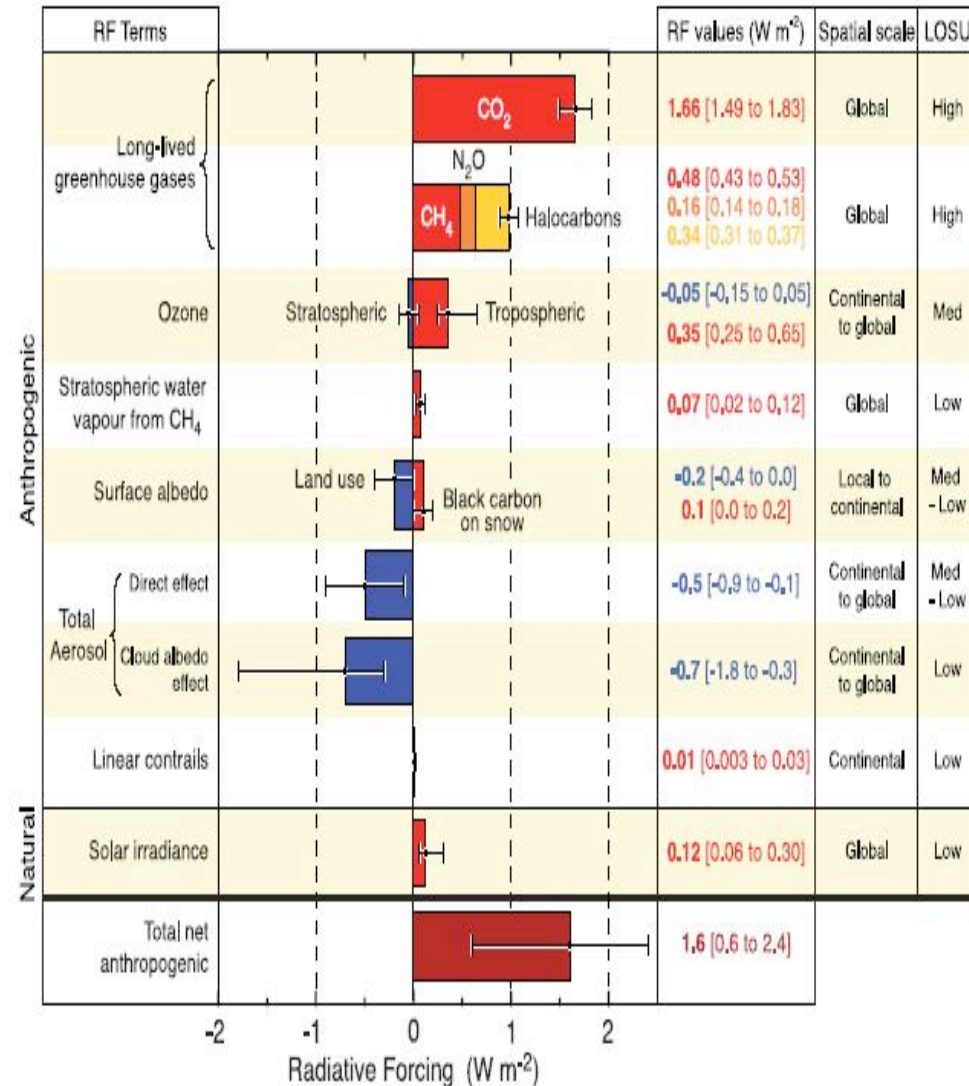
Atmospheric CO₂ at Mauna Loa Observatory



GHGs rise!

It's gotta do with us, at least a bit, ain't it?
But just how much?

RADIATIVE FORCING COMPONENTS



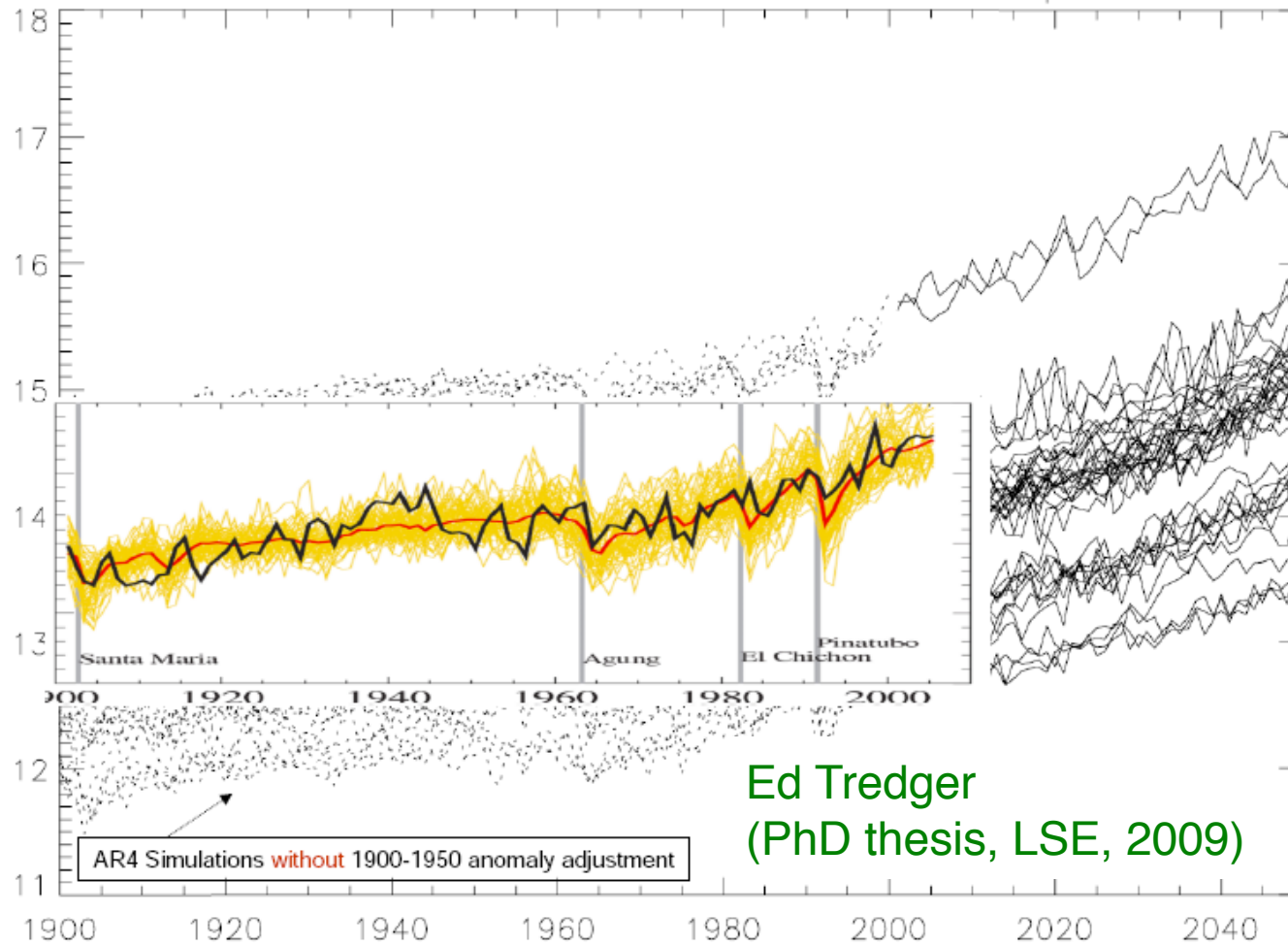
IPCC (2007)



AR4 adjustment of 20th century simulation

www.lseca

Hindcasts and Forecasts of Global Mean Temperature



Ed Tredger
(PhD thesis, LSE, 2009)



A tool for classification: stochastic equivalence

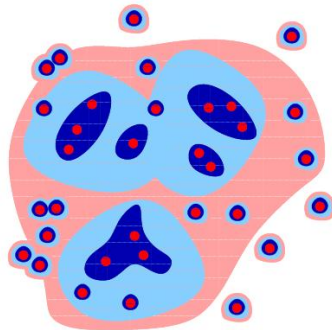
- **Stochastic equivalence:** two cocycles $\varphi_1(t, \omega)$ and $\varphi_2(t, \omega)$ are conjugated iff there exists a **random homeomorphism** $h \in \text{Homeo}(X)$ and an invariant set $\tilde{\Omega}$ of full \mathbb{P} -measure (w.r.t. θ) such that $h(\omega)(0) = 0$ and:

$$\varphi_1(t, \omega) = h(\theta(t)\omega)^{-1} \circ \varphi_2(t, \omega) \circ h(\omega); \quad (2)$$

h is also called **cohomology** of φ_1 and φ_2 . It is a **random change of variables!**

- **Motivation:** We would like to measure quantitatively as well as qualitatively the difference between **climate models**.

Stochastic equivalence - Could noise help the classification?



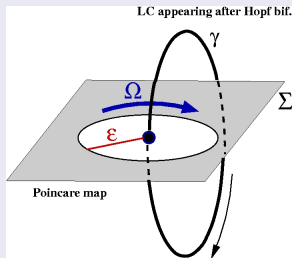
As the noise variance tends to zero and/or the parametrizations are switched off, one recovers the structural instability, as a “granularity” of model space. For **nonzero variance**, the random attractor $\{\mathcal{A}(\omega)\}$ associated with several GCMs might fall into **larger** and **larger** classes as the **noise level increases**.

Investigation of these ideas on a family of dynamical toy systems - **Theoretical and numerical results**

V. Arnold's family of diffeomorphisms

- We want to perform a *classification* in terms of **stochastic equivalence**.
- Our first **theoretical laboratory** is **Arnold's family of diffeomorphisms of the circle**:

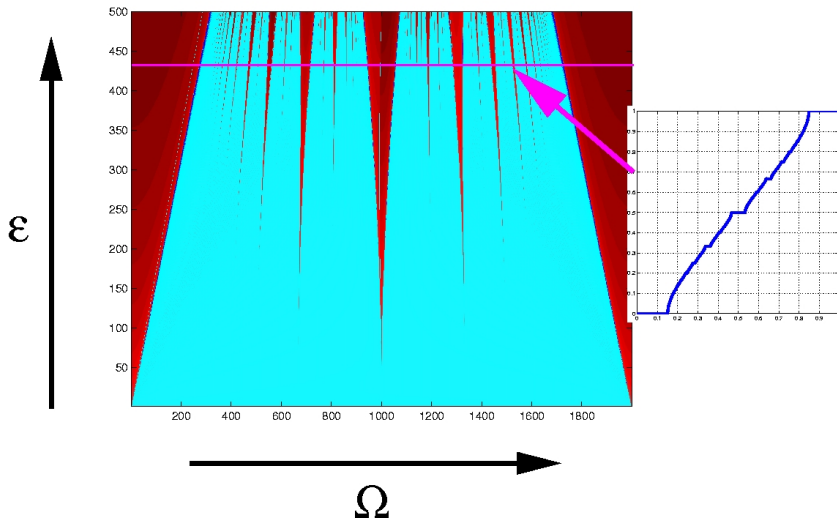
$$x_{n+1} = F_{\Omega, \varepsilon}(x_n) := x_n + \Omega - \varepsilon \sin(2\pi x_n) \pmod{1}$$



Which **paradigm** is represented by this family? Why this family?

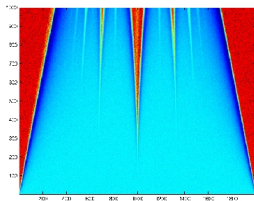
- **Frequency-locking phenomena & Devil's staircase**
- **Topological classification** of Arnold's family $\{F_{\Omega,\varepsilon}\}$:
 - **Countable** regions of **structural stability**,
 - **Uncountable structurally unstable systems** with **non-zero Lebesgue measure!**
- **Two types of attractors:**
 - **Periodic orbits** in the circle.
 - **The whole circle.**

Arnold's tongues and Devil's staircase

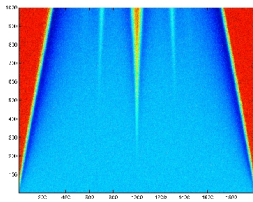


Effect of the noise on topological classification?

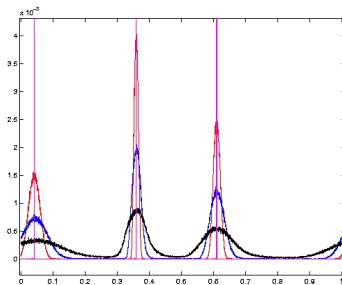
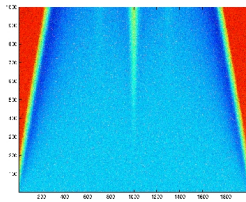
$\sigma=0.05$



$\sigma=0.10$



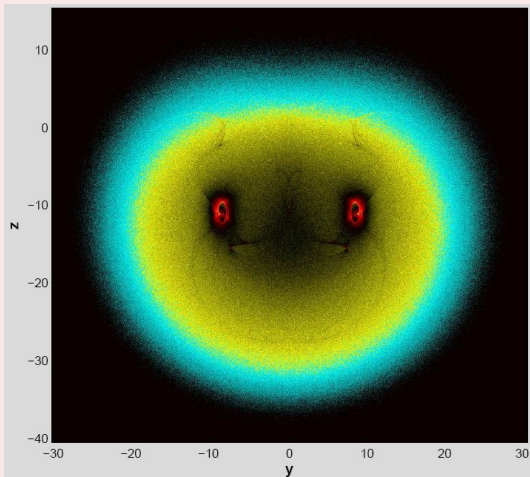
$\sigma=0.15$



Effect of the noise on the PDF of Arnold's tongue 1/3

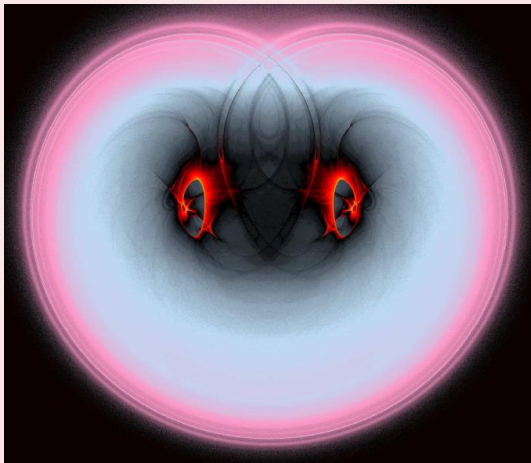
Sample measure supported by the R.A.

Another proj. of the sample measure, "friendlier"



- The next slides are similar, with different **noise level α** and more I.D....

Sample measure supported by the R.A.



- 1 Billion I.D., and a different color palette!
- Intensity is $\alpha = 0.2$.
- Do you want different noise intensities?

The Sinai-Ruelle-Bowen (SRB) property

- RDS theory offers a rigorous way to define random versions of stable and unstable manifolds, via the Lyapunov spectrum, the Oseledec multiplicative theorem, and a random version of the Hartman-Grobman theorem.
- When the sample measures μ_ω of an RDS have **absolutely continuous conditional measures** on the random unstable manifolds, then μ_ω is called a **random SRB measure**.
- If the sample measure of an RDS φ is SRB, then its a “physical” measure in the sense that:

$$\lim_{s \rightarrow -\infty} \frac{1}{t-s} \int_s^t G \circ \varphi(s, \theta_{-s}\omega)x \, ds = \int_{\mathcal{A}(\theta_t\omega)} G(x) \mu_{\theta_t\omega}(dx), \quad (3)$$

for almost every $x \in X$ (in the Lebesgue sense), and for every continuous **observable** $G : X \rightarrow \mathbb{R}$.

- The measure μ_ω is also the image of the Lebesgue measure under the stochastic flow φ : for each region of $\mathcal{A}(\omega)$, it gives the **probability to end up** on that region, when starting from a volume.

A remarkable theorem of Ledrappier and Young (1988)

- Ledrappier and Young have proved that, that if the **stationary solution**, ρ , of the **Fokker-Planck equation** associated to an SDE presenting a Lyapunov exponent > 0 , has a **density** w.r.t. the Lebesgue measure, then:

μ_ω is a random SRB measure.

- This theorem applies to a large class of dissipative stochastic systems, namely the hypoelliptic ones that exhibit a Lyapunov exponent > 0 : they all support a random SRB measure.
- Furthermore, we have the important relation:

$$\mathbb{E}(\mu_\bullet) = \rho, \quad (4)$$

where ρ is the stationary solution of the Fokker-Planck equation, when the latter is unique.

Climatic uncertainties & moral dilemmas



Thought leaders
Rice, top left, spoke of multilateralism, while Bono, left, demanded more action on poverty. Presidents Karzai and Musharraf, right, both face troubles at home

♥ ... keep today's climate for tomorrow?



Agitator Gore
wants a global compact to tackle climate change and poverty

♥ **Feed the world today or...**

Davos, Feb. 2008, photos by *TIME Magazine*, 11 Feb. '08;
see also Hillerbrand & Ghil, *Physica D*, 2008, **237**, 2132–2138,
[doi:10.1016/j.physd.2008.02.015](https://doi.org/10.1016/j.physd.2008.02.015) .

The Biofuel Myth

- ◆ Fine illustration of the moral dilemmas (*).
- ◆ Conclusion:
“**festina lentae,**”
as the Romans (**)
used to say..

(**) ~ Han dynasty

(*) Hillerbrand & Ghil, *Physica D*, 2008,
[doi:10.1016/j.physd.2008.02.015](https://doi.org/10.1016/j.physd.2008.02.015),
available on line.



Climate Change 1816–2008



T. Géricault, 1819,
Le Louvre

M. Gillot, 2008,
Le Monde

