Climate projections and pathwise linear response of climate to forcing

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1. Can we reduce uncertainties in climate projections?

- Last IPCC report (AR4): GCM overfitting over the 20th century leads to divergent behavior for the latter part of the 21st century.

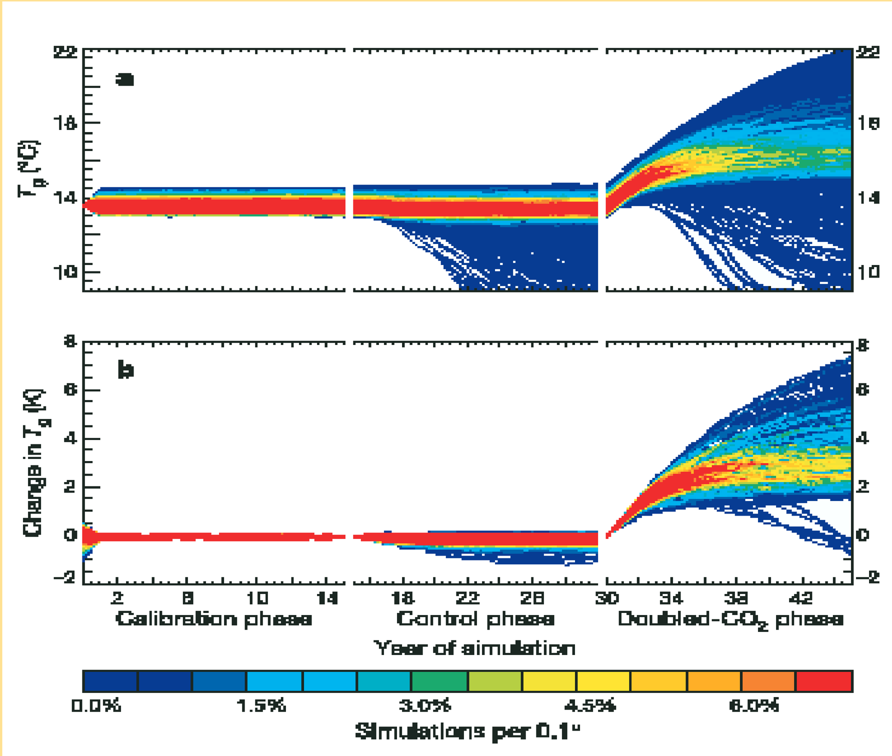


Figure 1: Frequency distribution of global mean, annual mean, near-surface temperature (T_g) for
(a) 2,017 GCM simulations, and doubled CO_2; and for
(b) a subset of 414 stable simulations, without substantial climate drift (Stainforth *et al.*, Nature, 2005).

- The range-of-uncertainty issue may be an intrinsic, rather than a mere "tuning" problem: we need theoretical insights, as well as checking theoretical results on intermediate climate models.

  Our analysis proceeds along the three following axes:

(1) Parameter sensitivity is investigated in an infinite-dimensional dynamical systems with delay, which models the El-Nino/Southern-Oscillation (ENSO) variability.
(2) Parameter sensitivity is explored in a stripped-down climate model, SPEEDY.
(3) Stochastic parametrizations are increasingly being used in modeling subgrid-scale processes in GCMs. We investigate the large-scale effect on random perturbations in the framework of random dynamical system (RDS) theory, fundamental aspects of climate sensitivity is discussed in the framework of Ruelle's response theory.

What are Random Dynamical Systems (RDSs)?

- We have a model of the noise $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ that is parametrized by time. The parametrization of realizations ω is provided by a **driving system** θ_t which is an **ergodic one-parameter group**.
- The dynamics is viewed on a "phase space $\times$ probability space," $X \times \Omega$, called the **bundle**, and the cocycle property enables one to treat trajectories as *flows* on this bundle.
- A path of the stochastic process corresponds to a selection of points in each fiber of the resulting bundle. Fibers are "glued together" by noise. The cocycle, also called RDS, provides a fiber-by-fiber view of the dynamics.

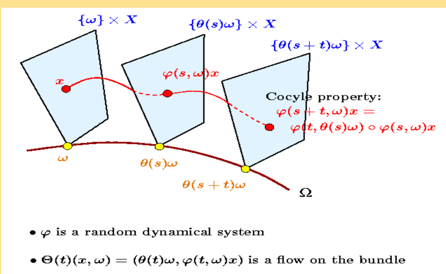


Figure 2: The bundle and the cocycle: the stochastic dynamics as a flow.

Random attractors and invariant measures

- We are looking for measures μ on $\Omega \times X$ that are invariant w.r.t. the dynamics. Central to our approach is the concept of sample measures μ_ω of μ. Mathematically, it is given by $\mu(B) = \int_\Omega \mu_\omega(B_\omega)\mathbb{P}(d\omega)$, with B_ω the intersection of a measurable set B of $X \times \Omega$ with an ω-fiber. Physically? Let's see.
- The random attractor, $\mathcal{A}(\omega)$ in Fig. 3, involves pullback attraction: we look at the phase-space location at time t starting several experiments far enough in the past and *for the same realization*. Hence we assess the "attracting regime" at time t.
- The sample measure μ_ω evolves with time, $\mu_\omega \mapsto \mu_{\theta_t\omega}$, and corresponds to the frozen statistics at time t: for each piece of $\mathcal{A}(\omega)$, it gives the probability to end up on that piece.

Schematic view of the random attractor's life

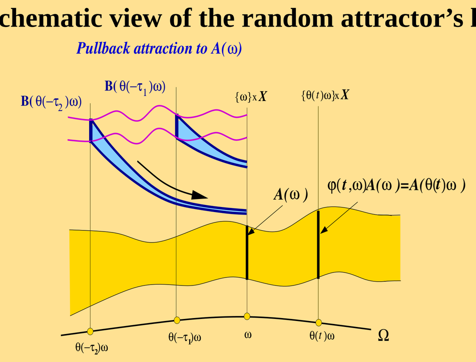


Figure 3: The curved arrow depicts the pullback attraction.

2. Sample measures: a change of paradigm

Effect of noise on the Lorenz system

- The Lorenz model is perturbed by multiplicative noise: $\mathrm{d}x = s(y-x)\mathrm{d}t + \sigma x \mathrm{d}W_t$, $\mathrm{d}y = (rx - y - xz)\mathrm{d}t + \sigma y \mathrm{d}W_t$, $\mathrm{d}z = (-bz + xy)\mathrm{d}t + \sigma z \mathrm{d}W_t$, where W_t is a Wiener process and $\sigma > 0$ the noise intensity; we call the resulting model **[SLM]**. The figure 4 corresponds to projection onto the (y,z) plane, $\int \mu_\omega(x,y,z)\mathrm{d}x$. One billion initial points have been used and the pullback attractor is computed for $t = 40$. The parameter values are the classical ones — $r = 28$, $s = 10$, and $b = 8/3$ — while $\sigma = 0.3$ and $\delta t = 5 \cdot 10^{-3}$. The color bar is on a log-scale and quantifies the probability to end up in a particular region of phase space.

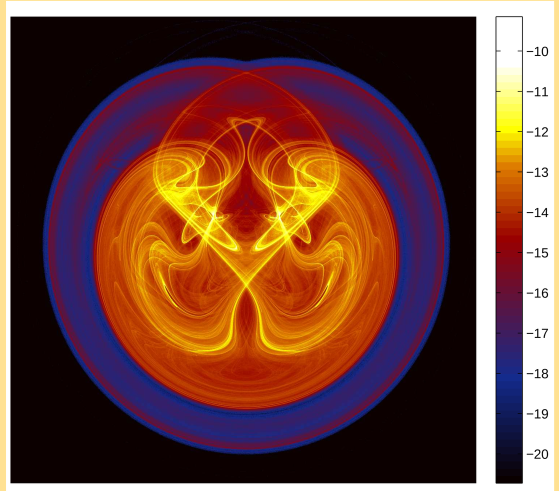


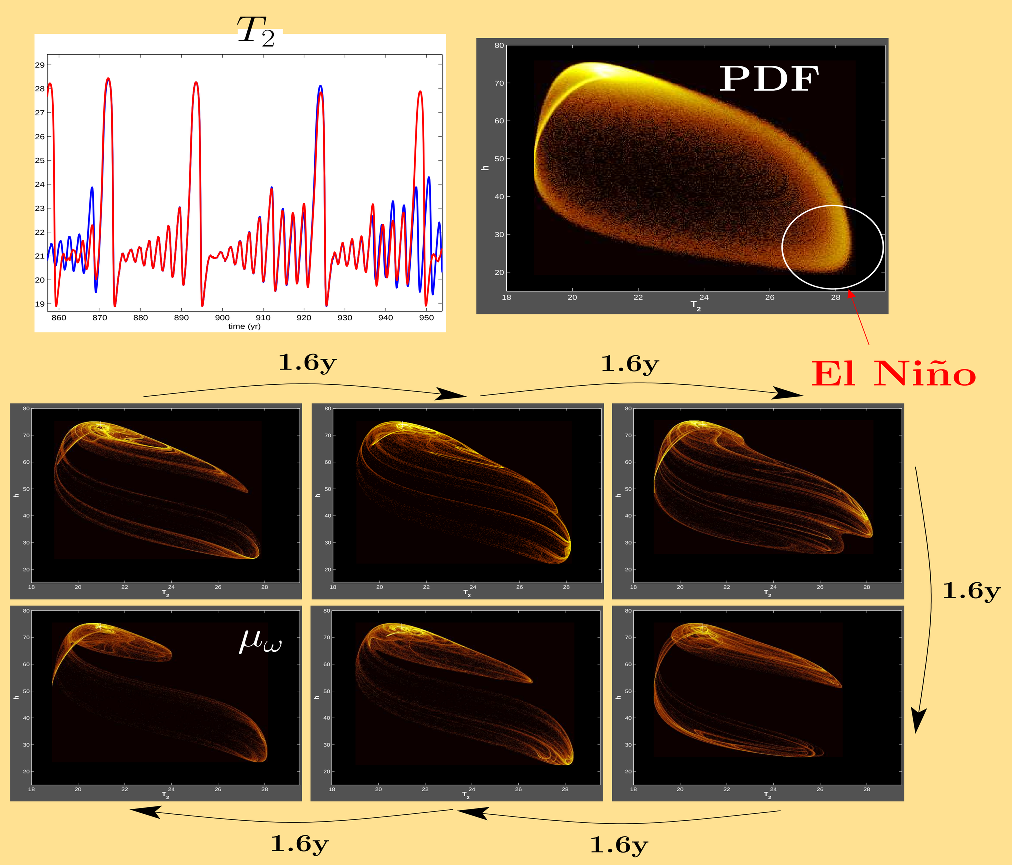
Figure 4: Snapshot of a stochastic Lorenz model's **global random attractor** $\mathcal{A}(\omega)$ and of the corresponding sample measure μ_ω, for a given, fixed realization ω. Notice the interlaced filament structures between highly (yellow) and moderately (red) populated regions.

Application to a stochastic El Niño model

- We consider a low-order, coupled tropical-atmosphere–ocean model of ENSO (Timmermann & Jin, *GRL*, 2002; hereafter **TJ model**). Three variables: thermocline depth anomaly h, and SSTs T_1 and T_2 in the western and eastern basin.

$$\begin{aligned}
\dot{T}_1 &= -\alpha(T_1 - T_r) - \tfrac{2\epsilon u}{L}(T_2 - T_1),\\
\dot{T}_2 &= -\alpha(T_2 - T_r) - \tfrac{w}{H_m}(T_2 - T_{sub}),\\
\dot{h} &= r(-h - bL\tau/2),\\[1em]
T_{sub} &= T_r - \tfrac{Tr - T_{r0}}{2}[1 - \tanh(H + h_2 - z_0)/h^*]\\
\tau &= \tfrac{a}{\beta}(T_1 - T_2)[\xi_t - 1].
\end{aligned}$$

The quantities are τ, the wind stress anomalies, equatorial upwelling $w = -\beta\tau/H_m$, zonal advection $u = \beta L\tau/2$, and subsurface temperature T_{sub}. Wind stress bursts are modeled as white noise ξ_t of variance σ and ϵ measures the strength of the zonal advection. We refer to **TJ** for the other parameters.



- Intermittency is illustrated in the upper-left panel, for two different initial states at $t = 0$ (blue and red curves) and the same realization ω; only the red curve appears where the two are visually indistinguishable. Six snapshots of the attractor and the sample measures μ_ω they support are shown at regular, 1.6-year intervals in the bottom panels; they are projected onto the $(h - T_2)$ plane, with T_2 on the abscissa, and their timing corresponds to interannual variability. The forward PDF is shown in the upper-right panel: it averages the sample measures μ_ω, *i.e.* "$\mathbb{E}(\mu_\omega)$ =PDF."
- Intermittency is synonymous of "weak chaos" in stochastic system and has been observed in other empirical models of ENSO; cf poster Kondrashov *et al.*.

3. Low frequency variability (LFV) and sample measures

- Two types of motion are present in the evolution of the sample measures $\mu_{\theta_t\omega}$. First, a pervasive "jiggling" of the overall structure can be traced back to the roughness of the Wiener process and to the multiplicative way it enters into the [SLM] model. Second, there is a smooth and quite regular low-frequency motion present in the evolution of the sample measures, which seems to be driven by the deterministic system's unstable limit cycles and is thus related to the well-known lobe dynamics. The latter motion is illustrated in Fig. 6.

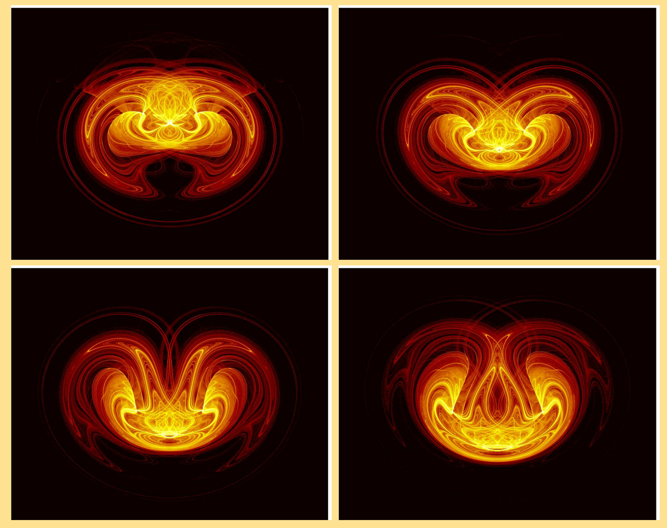


Figure 6: The time interval δt between two consecutive snapshots of the sample measure – moving from left to right and top to bottom – is related to the LFV of the [SLM] model.

- The sample measures give all the frozen statistics of a stochastic system (cf. last panel) and are furthermore naturally linked with the LFV of the system.

4. Pathwise sensitivity of sample measures

Pathwise statistical stability in a stochastic Lorenz model

- We keep the same parameter values as in Fig. 4 and perturb slightly the noise intensity σ from its value $\sigma_0 = 0.5$. A cube C is fixed in $\mathbb{R}^3$ such that the support of the sample measures lie always in C, and C is discretized over a regular mesh with N^3 nodes: we obtain then a numerical approximation $\mu_\omega^{\sigma_0,N}$ of the measures $\mu_\omega^{\sigma_0}$.

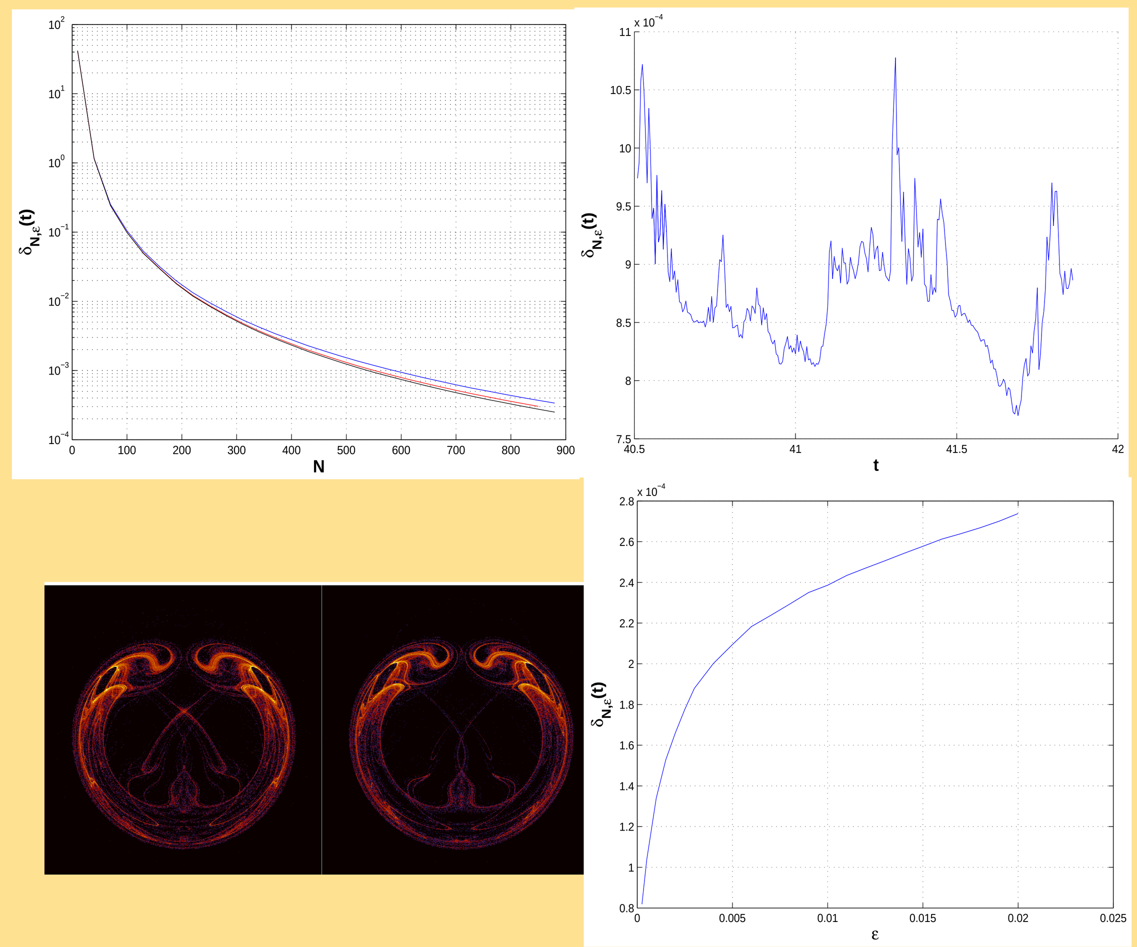


Figure 7: Weak pathwise sensitivity of the sample measure of the [SLM].

- The upper-left panel shows the L_1-error $\delta_{N,\epsilon}(t)$ as a function of N where $\delta_{N,\epsilon}(t) := \int_C |p_{\theta_t\omega}^{\sigma_0,N} - p_{\theta_t\omega}^{\sigma_0-\epsilon,N}|\,\mathbf{dx}$, and where $p_{\theta_t\omega}^{\sigma,N}(\mathbf{x})$ is the "density" of the discrete sample measure $\mu_{\theta_t\omega}^{\sigma_0,N}$ for $\epsilon = 10^{-2}$ and three different sets of initial data; the number of points n in the latter is $n = 50^3, 100^3$ and 300^3 for the blue, red and black curves, respectively. The upper-right panel displays $\delta_{N,\epsilon}(t)$ for t varying over 1.5 time units, $40.5 < t < 42.0$, while $N = 600$ and $\epsilon = 10^{-2}$. The lower-left panel plots a snapshot of the two sample measures that correspond to noise intensity σ_0 and $\sigma_0 - \epsilon$ at the end of the time series of $\delta_{N,\epsilon}(t)$ in the upper-right panel. The lower-right panel shows $\delta_{N,\epsilon}(t)$ as a function of ϵ for $N = 900$ and $n = 200^3$; this error clearly converges to zero as $\epsilon \longrightarrow 0 \Rightarrow$ Statistical stability!
- Statistical stability means that the sample measure varies smoothly w.r.t. parameters.
- We have even a pathwise linear response as it shown clearly the lower-right panel of Fig. 5 for $\epsilon > 5.10^{-3} \Rightarrow$ no sudden changes in response to a small parameter change.

5. Mathematics of climate sensitivity

- How typical is it to encounter sudden changes in response to a small parameter change?
- Most of the results are based on ensemble of simulations.
- We propose here fundamental aspects of this question.

The Sinai-Ruelle-Bowen (SRB) property

- RDS theory offers a rigorous way to define random versions of stable and unstable manifolds, via the Lyapunov spectrum and the Oseledec multiplicative theorem.
- When the sample measures μ_ω of an RDS have absolutely continuous conditional measures on the random unstable manifolds, then μ_ω is called a *random SRB measure*.
- If the sample measure of an RDS φ is SRB, then its a "physical" measure in the sense that:

$$\lim_{s \to -\infty} \frac{1}{t-s} \int_s^t \Phi \circ \varphi(s, \theta_{-s}\omega)x \,\mathrm{d}s = \int_{\mathcal{A}(\theta_t\omega)} \Phi(x)\mu_{\theta_t\omega}(\mathrm{d}x), \qquad (1)$$

  for almost every $x \in X$ (in the Lebesgue sense), and for every continuous observable $\Phi : X \to \mathbb{R}$.
- These SRB measures govern in fact all the frozen statistics of the system.
- When SRB, the measure μ_ω is also the image of the Lebesgue measure under the stochastic flow φ: for each region of $\mathcal{A}(\omega)$, it gives the probability to end up on that region, when starting from a volume.
- It appears that most of stochastic systems exhibiting a positive Lyapunov exponent exhibit an random SRB measure (Ledrappier and Young, 1988). The stochastic Lorenz system as well as the stochastic model of ENSO used here satisfy this property. Figures 4 and 5 show therefore random SRB measures.

The Ruelle response formula

- From a mathematical point of view, climate sensitivity can be analyzed in terms of sensitivity of SRB measures.
- The thermodynamic formalism *à la* Ruelle, in the RDS context, helps to understand the response of systems out-of-equilibrium, to changes in the parameterizations.
- The Ruelle response formula: Given an SRB measure μ of an autonomous chaotic system $\dot{x} = F(x)$, an observable $\Phi : X \to \mathbb{R}$, and a smooth time-dependent perturbation G_t, then the time-dependent variations $\delta_t\mu$, of μ is given by:

$$\delta_t < \mu, \Phi >= \int_{-\infty}^{t} d\tau \int \mu(dx) G_\tau(x) \cdot \nabla_x(\Phi \circ \varphi_{t-\tau}(x)), \qquad (2)$$

  where φ_t is the flow of the unperturbed system $\dot{x} = F(x)$, and $< \mu, \Phi >:= \int \Phi(x)\mu(dx)$.
- This formula permits to compute the response of the system without ensemble of long-run simulations. The invariant measure μ of the unperturbed system is the "expansive" part of the computation, but when μ is known the formula gives the response without integrating the perturbed system $\dot{x} = F(x) + G_t(x)$.

The susceptibility function

- In the case of a perturbation $G_t(x) := \phi(t)G(x)$, the Ruelle response formula can be written:

$$\delta_t < \mu, \Phi >= \int dt' \kappa(t - t')\phi(t'),$$

  where κ is called the response function. The Fourier transform $\mathcal{F}(\kappa)$ of the response function is called the susceptibility function.
- In this case $\mathcal{F}(\delta_t < \mu, \Phi >)(\xi) = \mathcal{F}(\kappa)(\xi)\mathcal{F}(\phi)(\xi)$ and since the r.h.s. is a product, there are no frequencies in the linear response that are not present in the signal.
- In general, the situation can be more complicated and the theory gives the following criteria of high-sensitivity:

  $\mathfrak{C}$: **Poles of the susceptibility function $\mathcal{F}(\kappa)(\xi)$ in the upper-half plane $\Rightarrow$ High sensitivity of the systems response function $\kappa(t)$.**
- Sudden changes in response to a small parameter change are thus related to the presence of such poles.
- The stochastic Lorenz system studied here do not exhibit such poles due to pathwise linear response, but other low-dimensional models are known to exhibit such poles as the Henon map.
- RDS theory offers a path for extending this criteria when random perturbations are considered.
- Sensitivity analysis of **stochastic intermediate climate models** within this framework is the next step!

References:

- Chekroun, M.D., E. Simonnet, and M. Ghil, 2010: Stochastic climate dynamics: Random attractors and time-dependent invariant measures, submitted.
- Ghil, M., M.D. Chekroun, and E. Simonnet, 2008: Climate dynamics and fluid mechanics: Natural variability and related uncertainties, *Physica D*, **237**, 2111–2126.