NP1.1. Mandelbrot Memorial Session Vienna, 6 March 2011
EGU 2011

Fractal Objects in Geoscience Models

Michael Ghil
Ecole Normale Supérieure, Paris, &
University of California, Los Angeles

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Pls. see these sites for further info.

http://www.atmos.ucla.edu/tcd/ (TCD);
http://www.environnement.ens.fr/(CERES); http://e2c2.ipsl.jussieu.fr/ (E2C2)

Motivation:
The smooth and the rough

- The late 19th and early 20th century saw a flourishing of both
  - continuous function theory (Riemann–integration, Weierstrass–approximation, etc.), and
  - measure theory (Borel–sets, Cantor–transfinite numbers, Daniell–integration, Lebesgue–measure)
- Physical applications of mathematics, though, tended to use differential equations, both ordinary and partial (ODEs and PDEs).
- These seemed to require the use of smooth, continuously differentiable functions for their solutions.
- It is only later in the 20th century that functional analysis and distributions were introduced to deal with rough solutions of PDEs (Friedrichs, Leray, Sobolev, Schwartz, etc.).

Outline

A. Some early self-similar objects in mathematics

– Pascal's triangle, the Cantor set, Peano curves

B. Self-similarity in classical DDS

– strange attractors

of maps (Hénon, Lozi) and flows (Lorenz)

C. Some novel math objects with scale invariance

– Boolean delay equations (BDEs)

– random dynamical systems (RDSs)

D. Conclusions and bibliography

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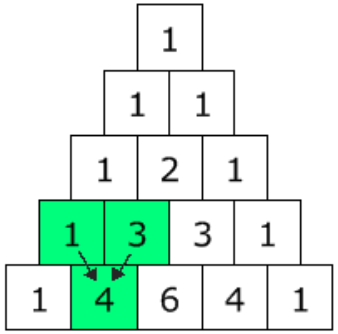
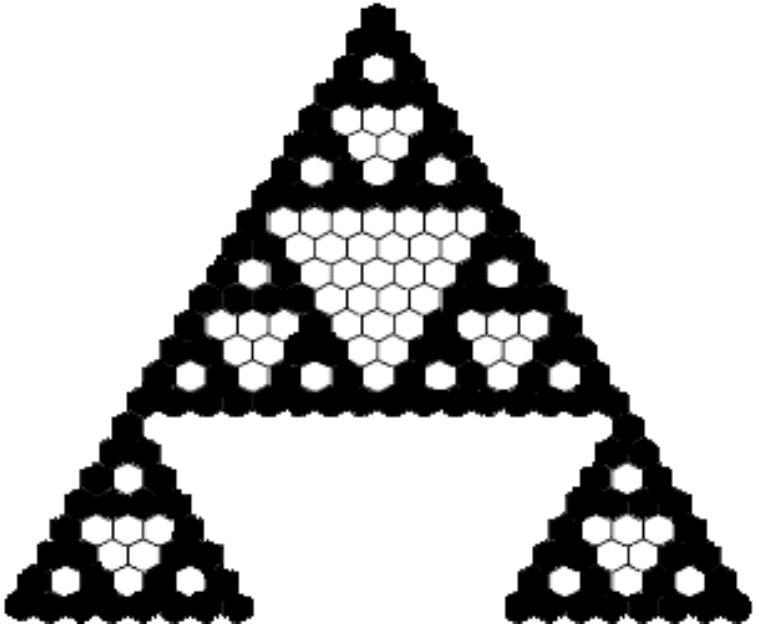
Early self-similar objects – I

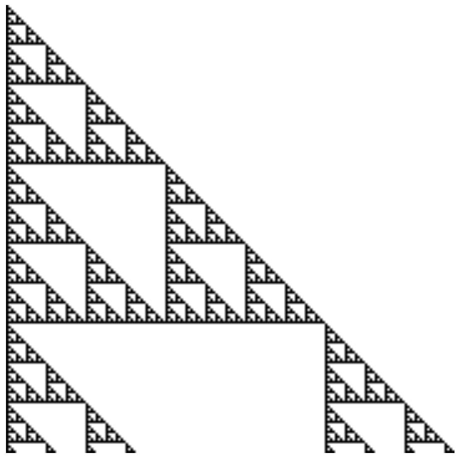
Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1



Fill in odd numbers in the triangle
In black, and you get the
"Sierpinski gasket"





B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman & Co, SF, 1982

Early self-similar objects – II

The Cantor (1883) ternary set [*]

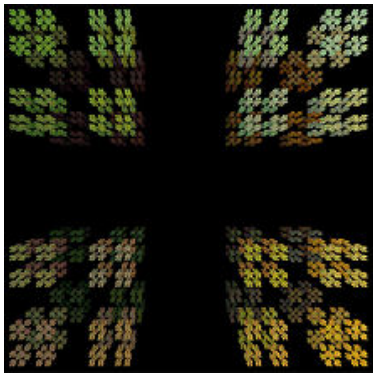
Recipe: remove middle thirds of remaining intervals

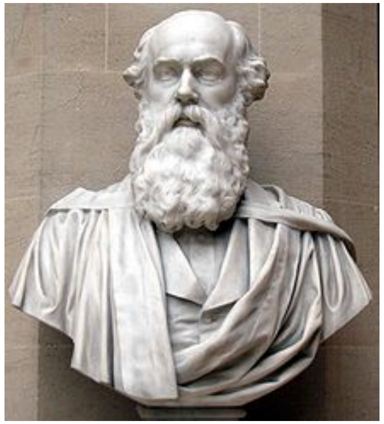
Measure = 0, Cardinality = Continuum,

Topology = Perfect set that is nowhere dense

Cantor dust

in 2-D and 3-D





H.J.S. Smith,
Oxford U. Museum

[*] H.J.S. Smith (1875), P. du Bois-Reymond (1880), V. Volterra (1881)

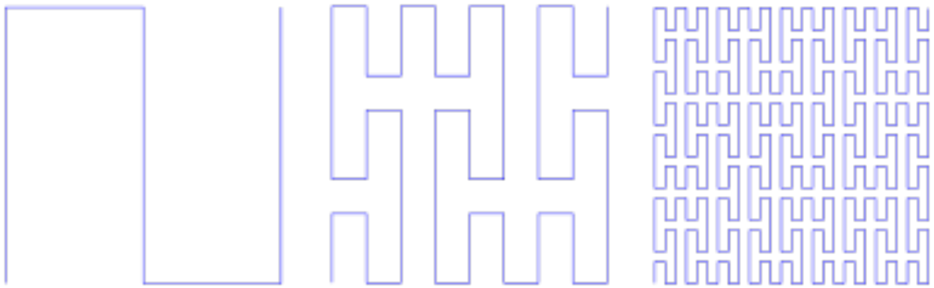
Early self-similar objects – III

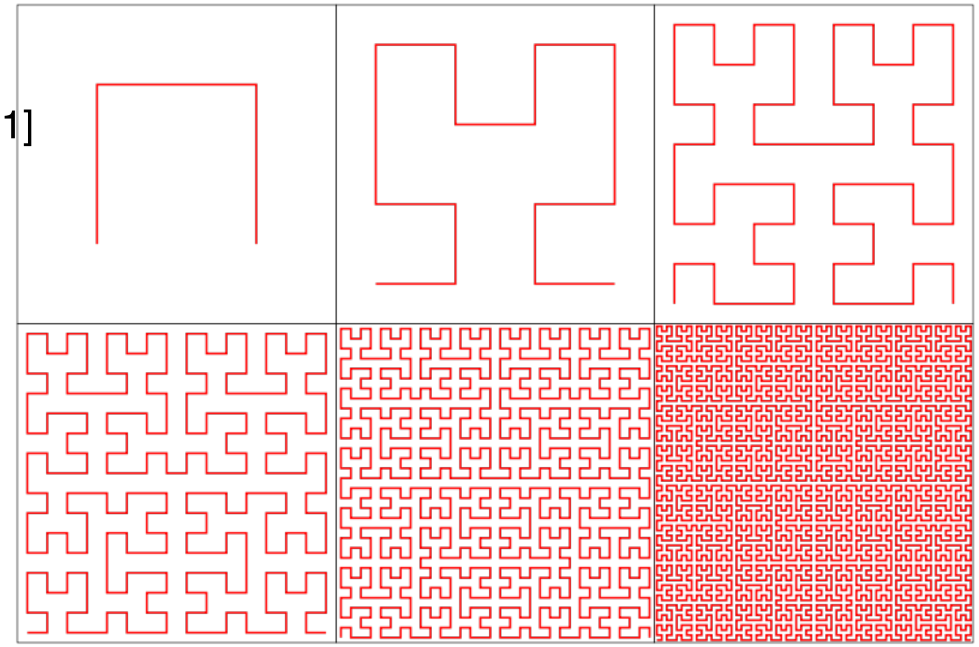
The Peano curve and other space-filling curves

Question: Is the cardinality of the unit interval and the unit square the same?

The first 3 steps of constructing the Peano (1890) curve +

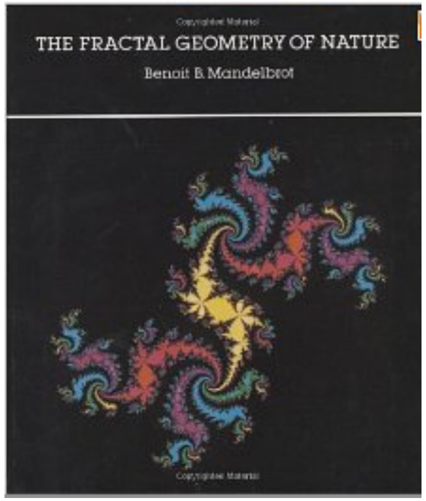
The first 6 steps of constructing the Hilbert (1891) curve





Typically, piecewise-linear constructions:

A curve (with endpoints) is a continuous function whose domain is the unit interval [0, 1] (C. Jordan, 1887)



Also Julia sets, etc.

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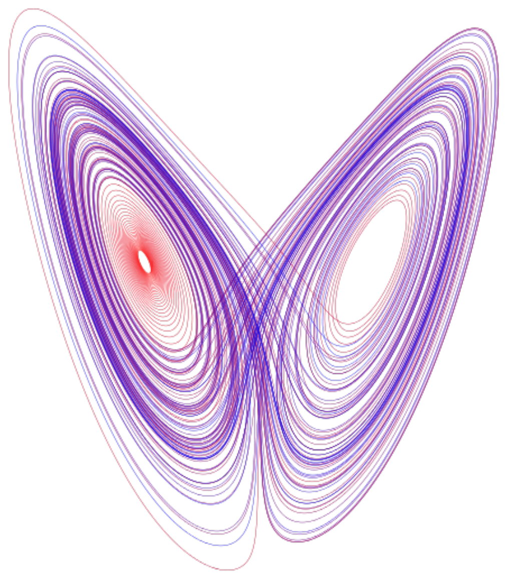
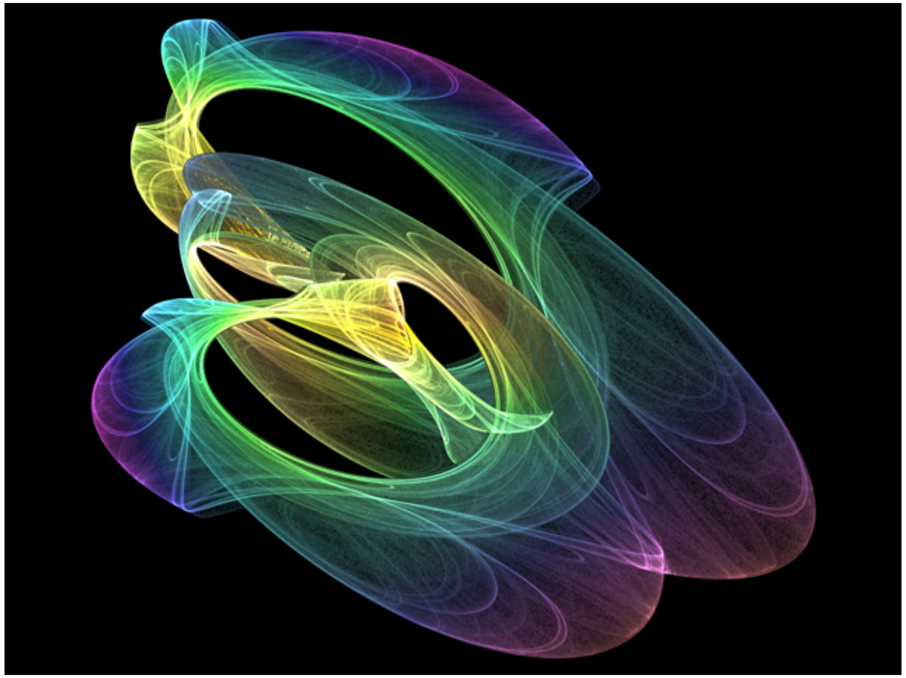
Self-similarity in classical DDS - I

The Lorenz attractor

Drilling through the butterfly's wings
yields a generalized Cantor set



Axonometric projection

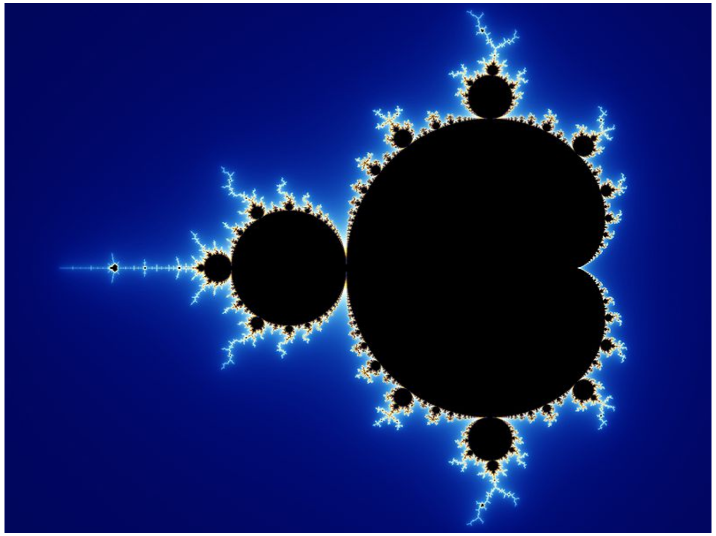


3-D rendering

Self-similarity in classical DDS - II

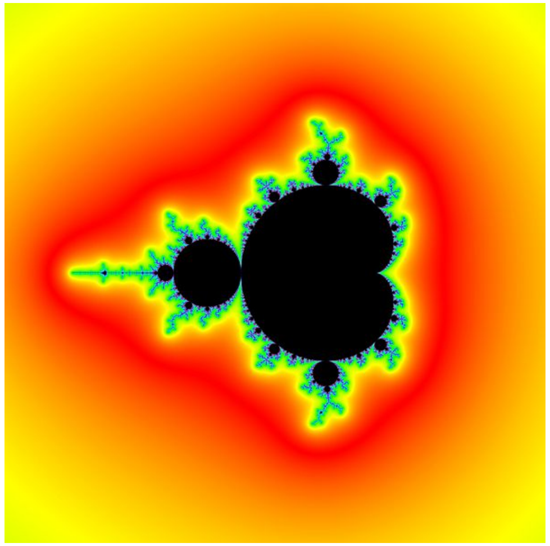
The Mandelbrot set is the set of values of $c \in \mathbb{C}$ for which the orbit of 0 under iteration of the complex quadratic polynomial $z_{n+1} = z_n^2 + c$ remains bounded as $n \to \infty$.

It is a compact set contained in the closed disk of radius 2 around the origin; it is connected.



Initial image of a Mandelbrot set zoom sequence

The M-set and the logististic map



Exterior distance estimate

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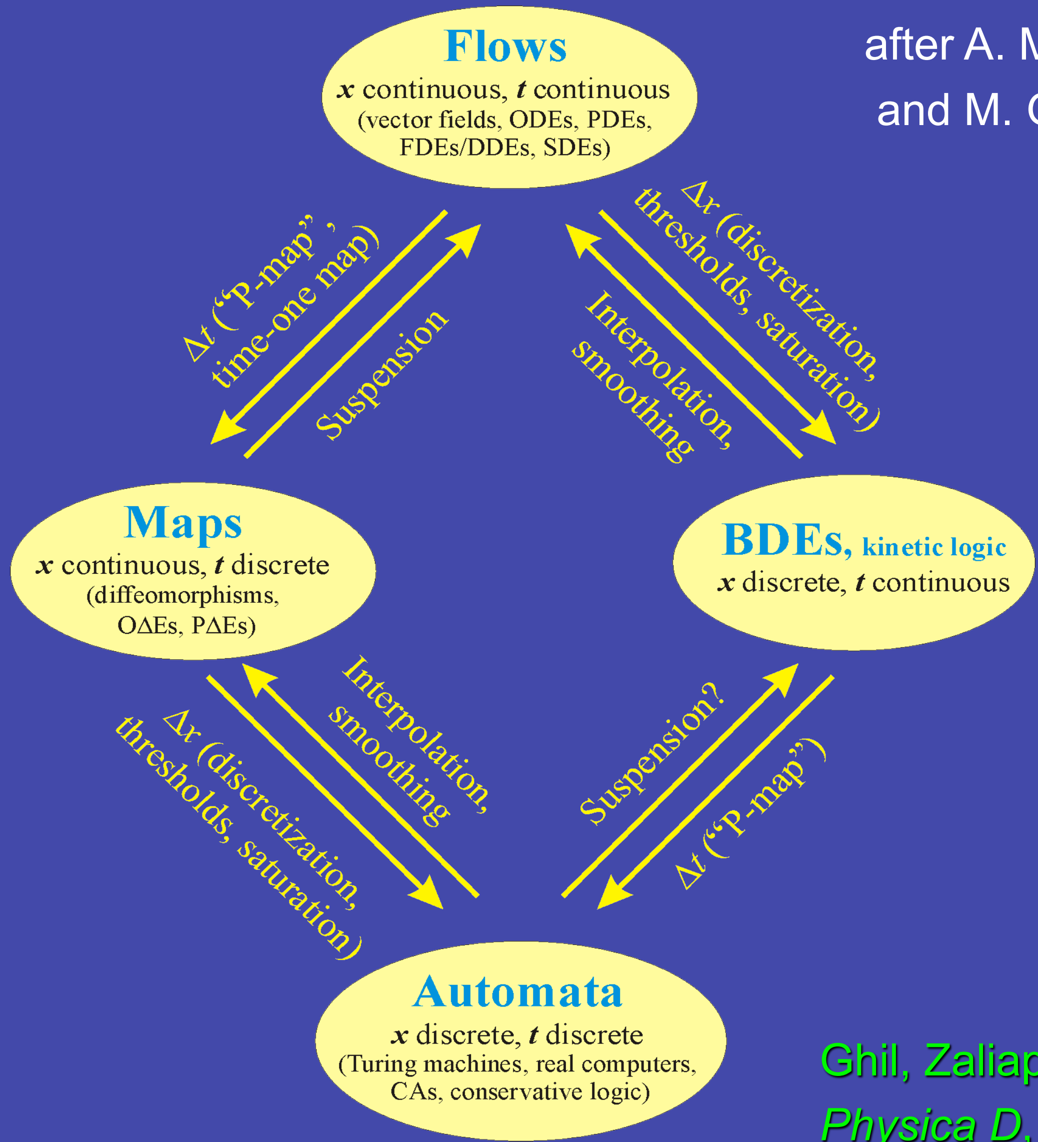
•The place of BDEs in dynamical systems theory

after A. Mullhaupt (1984)
and M. Ghil *et al*. (2008)



Ghil, Zaliapin & Coluzzi, 2008:
Physica D, **237**, 2967–2986.

What are BDEs?

Short answer:

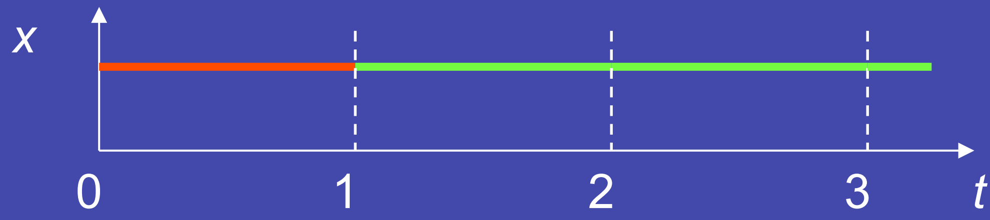
Maximum simplification of nonlinear dynamics
(non-differentiable time-continuous dynamical system)

Longer answer:

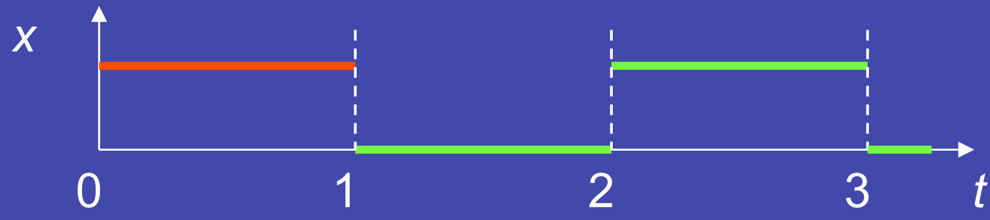
1) $x \in B = \{0, 1\}$

$x(t) = x(t-1)$

(simplest EBM: $x = T$)



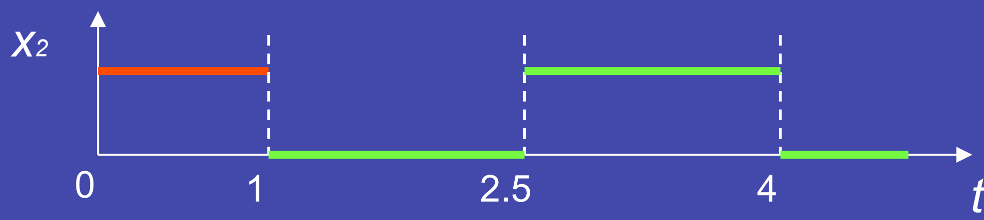
2) $x(t) = \overline{x}(t-1)$



3) $x_1, x_2 \in B = \{0, 1\}; 0 < \theta \leq 1$

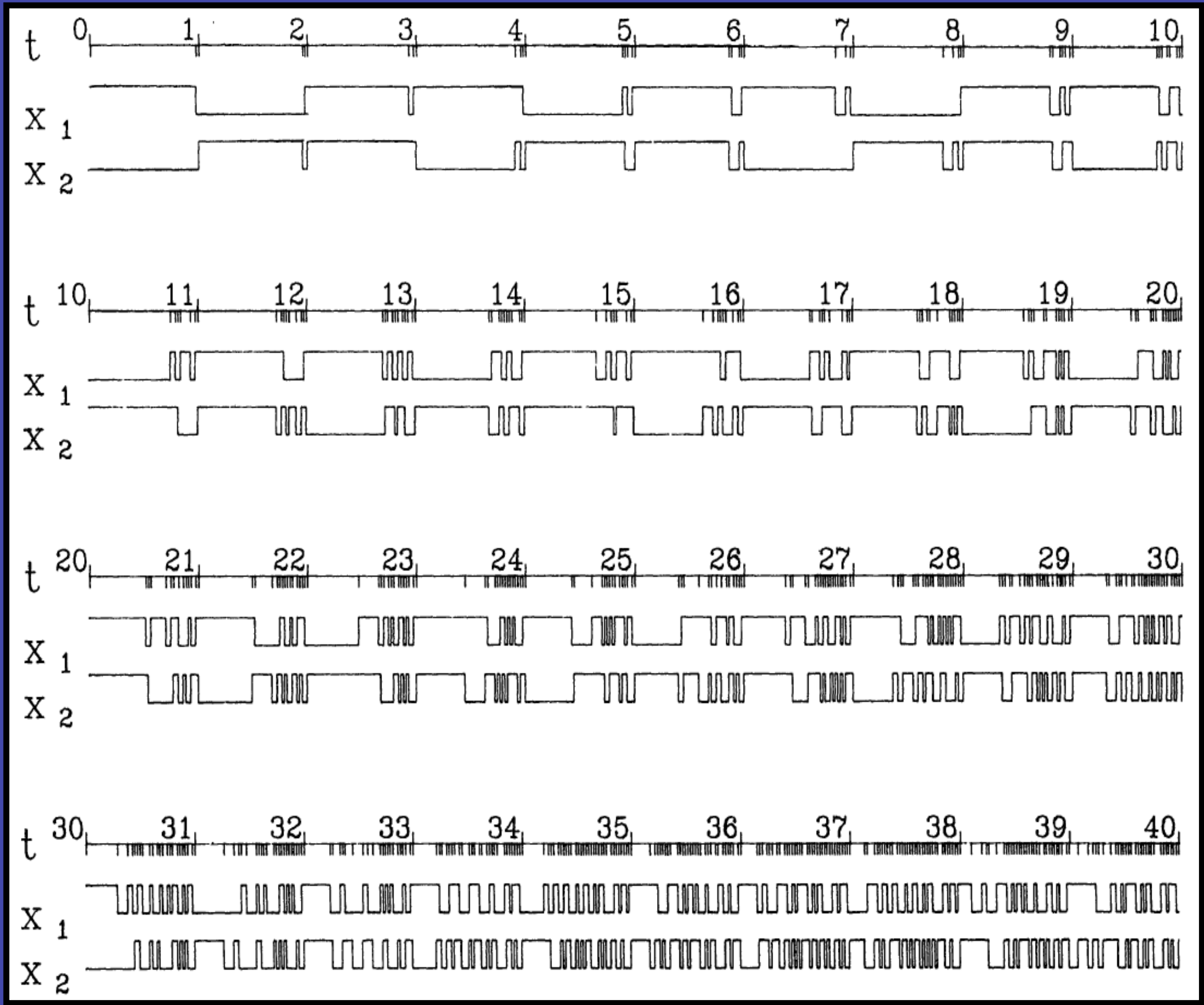
$$\begin{cases} x_1(t) = x_2(t-\theta), \theta = 1/2 \\ x_2(t) = \overline{x}_1(t-1) \end{cases}$$

Eventually periodic with
a period $= 2(1+\theta)$
(simplest OCM: $x_1 = m$, $x_2 = T$)



$$\begin{cases} x_1(t) = x_2(t-\theta) \\ x_2(t) = x_2(t-1)\nabla x_1(t-\theta) \end{cases}$$

θ is irrational



Increase in complexity!

Evolution: biological, cosmogonic, historical

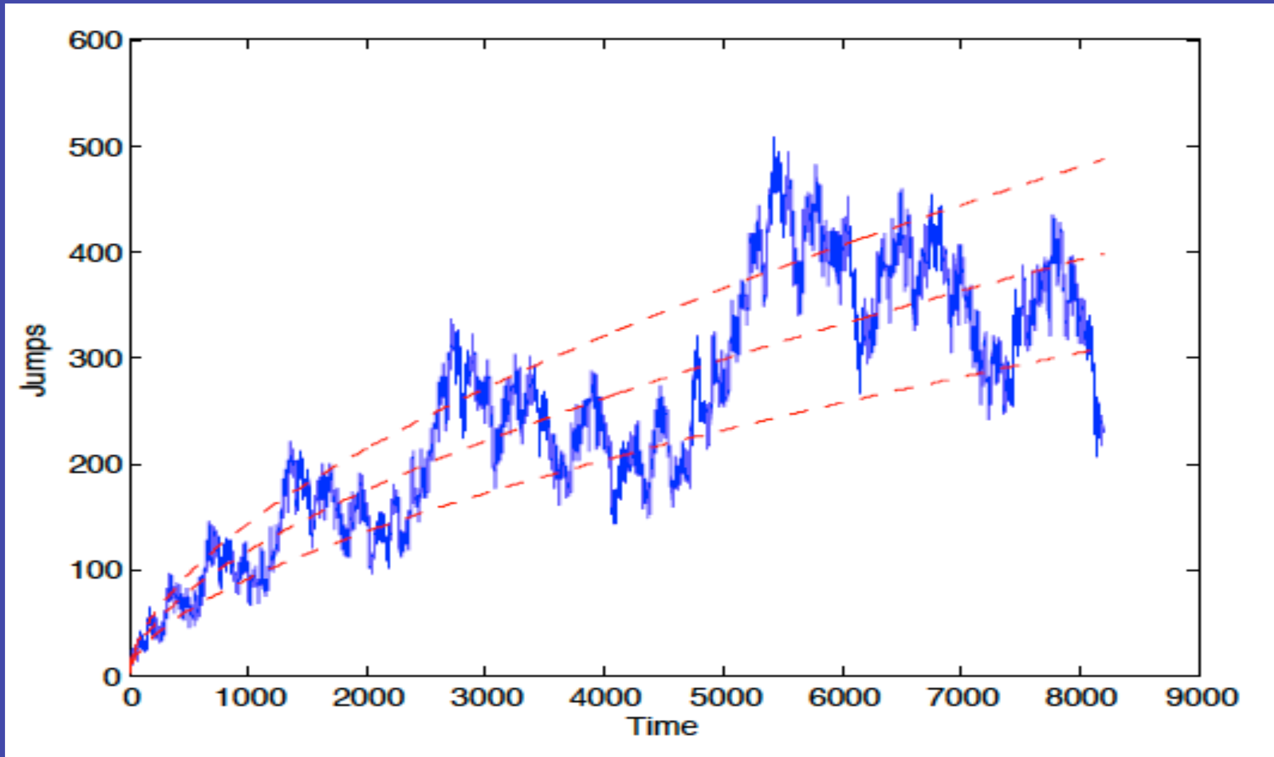
But how much?

Dee & Ghil, *SIAM J. Appl. Math*. (1984), **44**, 111-126

Aperiodic solutions with *increasing complexity*

$$x(t) = x(t-1)\nabla x(t-\theta), \qquad \theta = \frac{\sqrt{5}-1}{2} = \text{"golden ratio"}$$

Jump Function



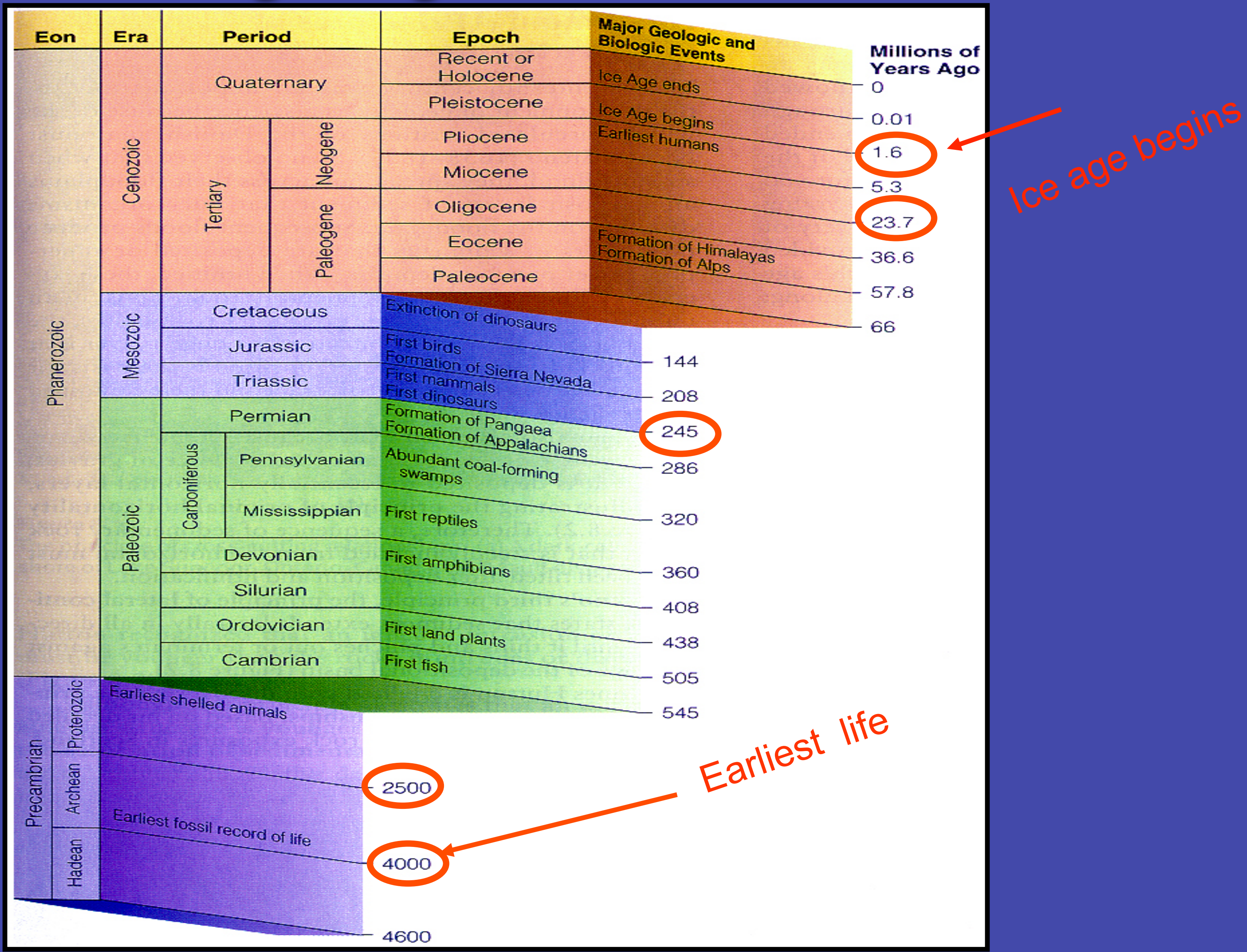
Time

Theorem:

Conservative BDEs with irrational delays have aperiodic solutions with a *power-law increase in complexity.*

N.B. Log-periodic behavior!

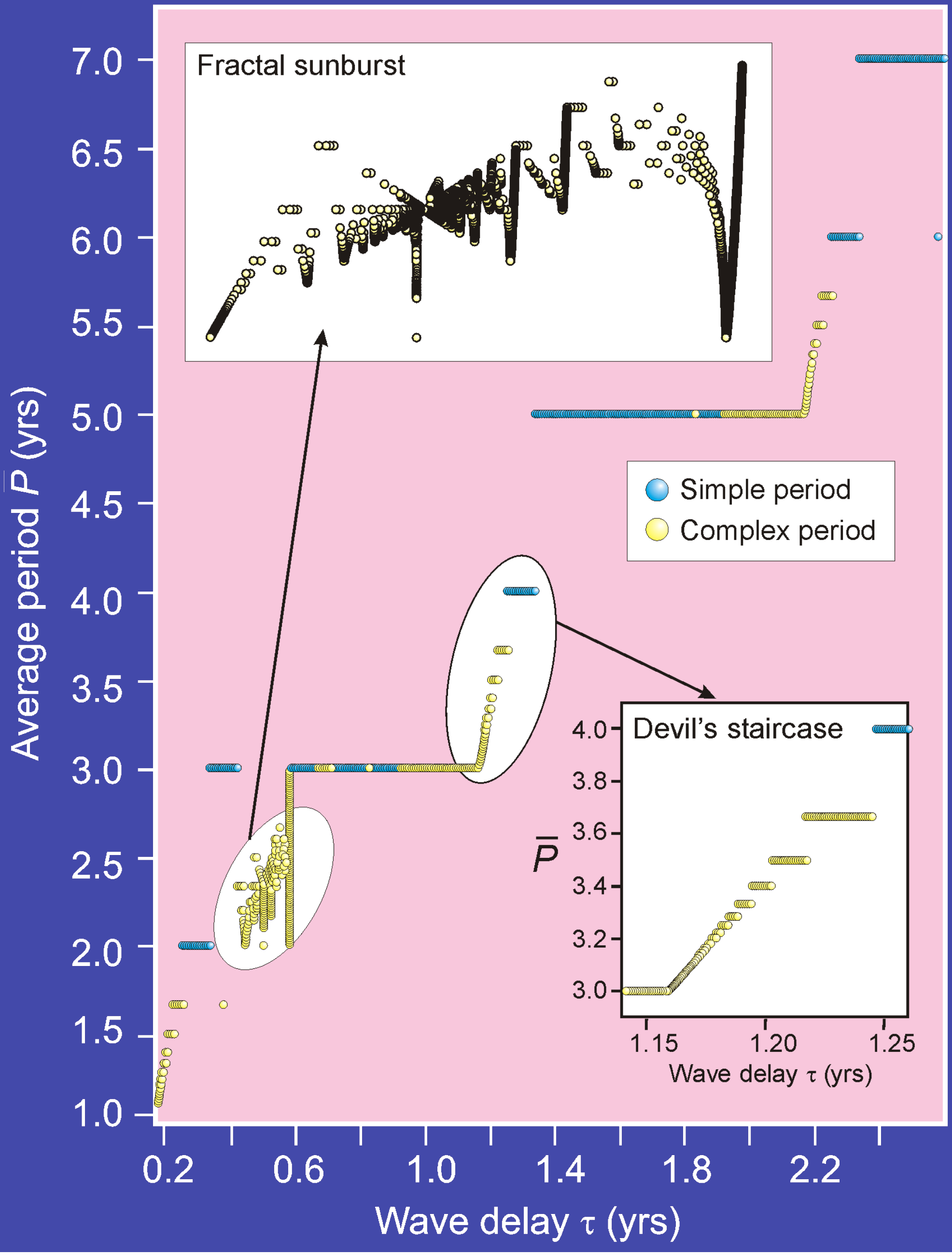
The geological time scale



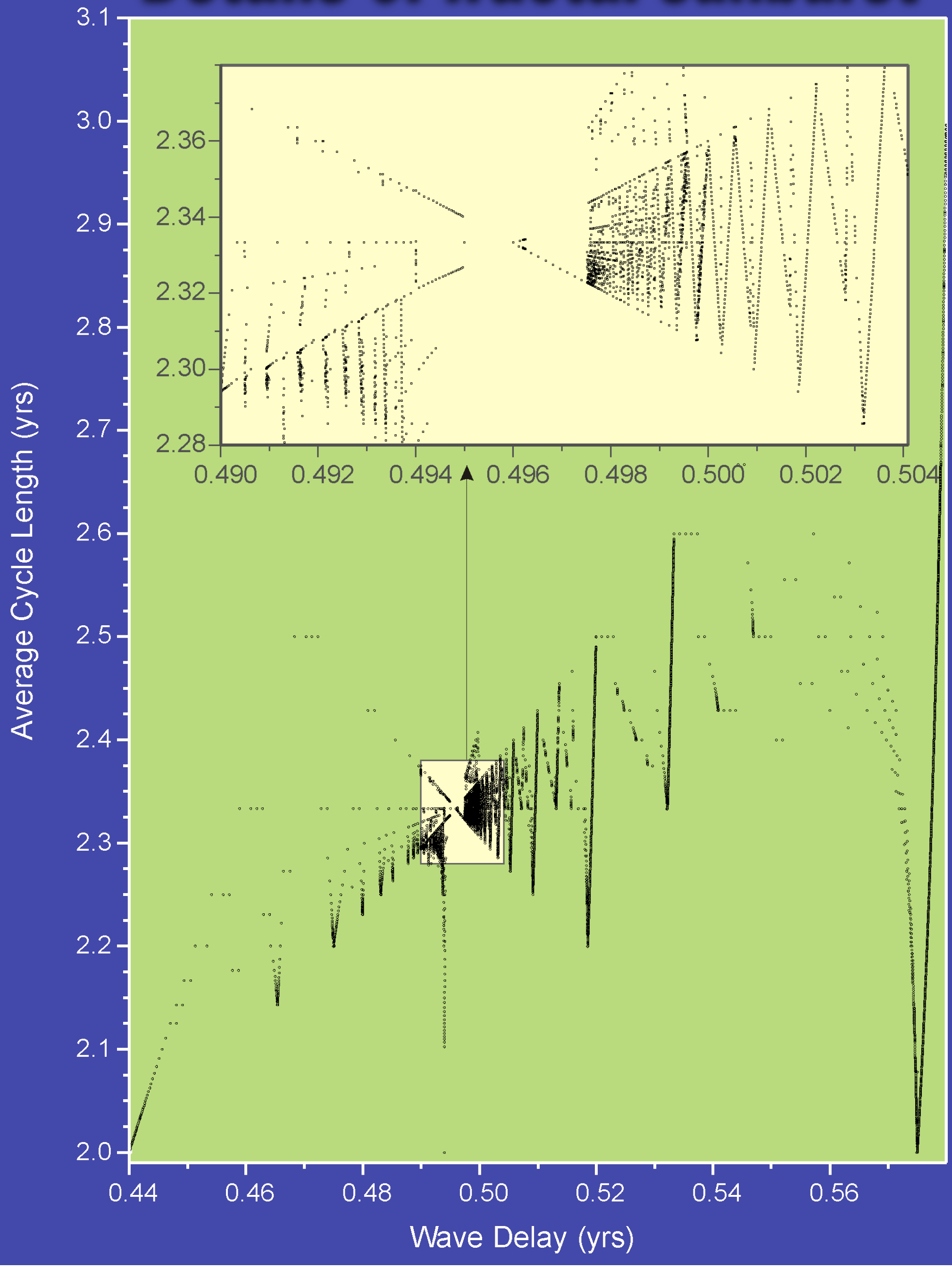
http://www.yorku.ca/esse/veo/earth/image/1-2-2.JPG

Density of events $\cong \log(t)$

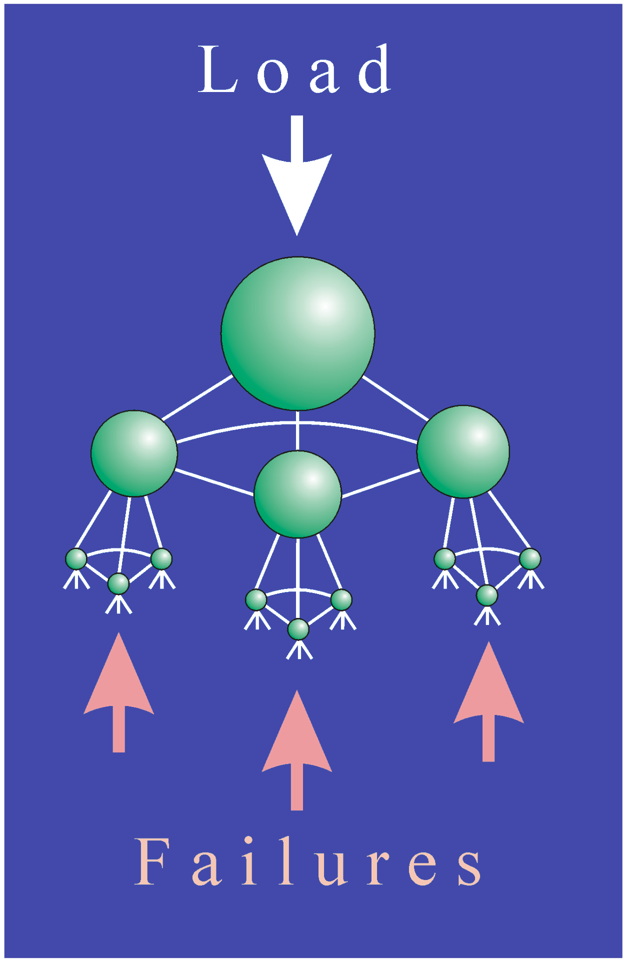
Devil's staircase and fractal sunburst



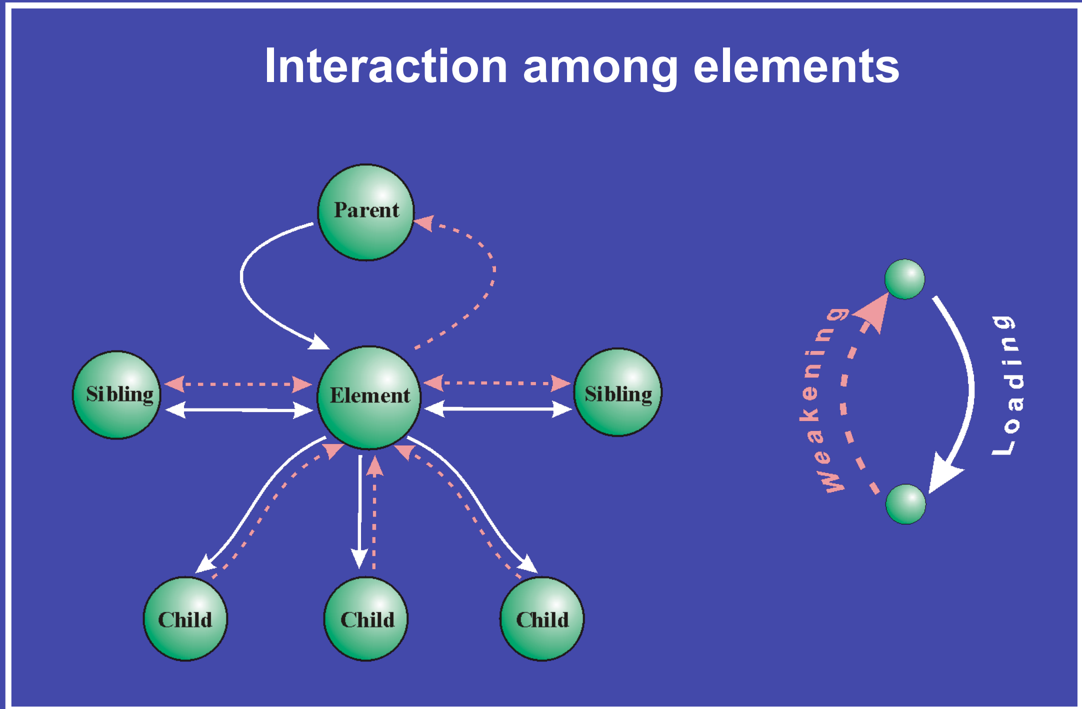
Details of fractal sunburst



Colliding-Cascade Model

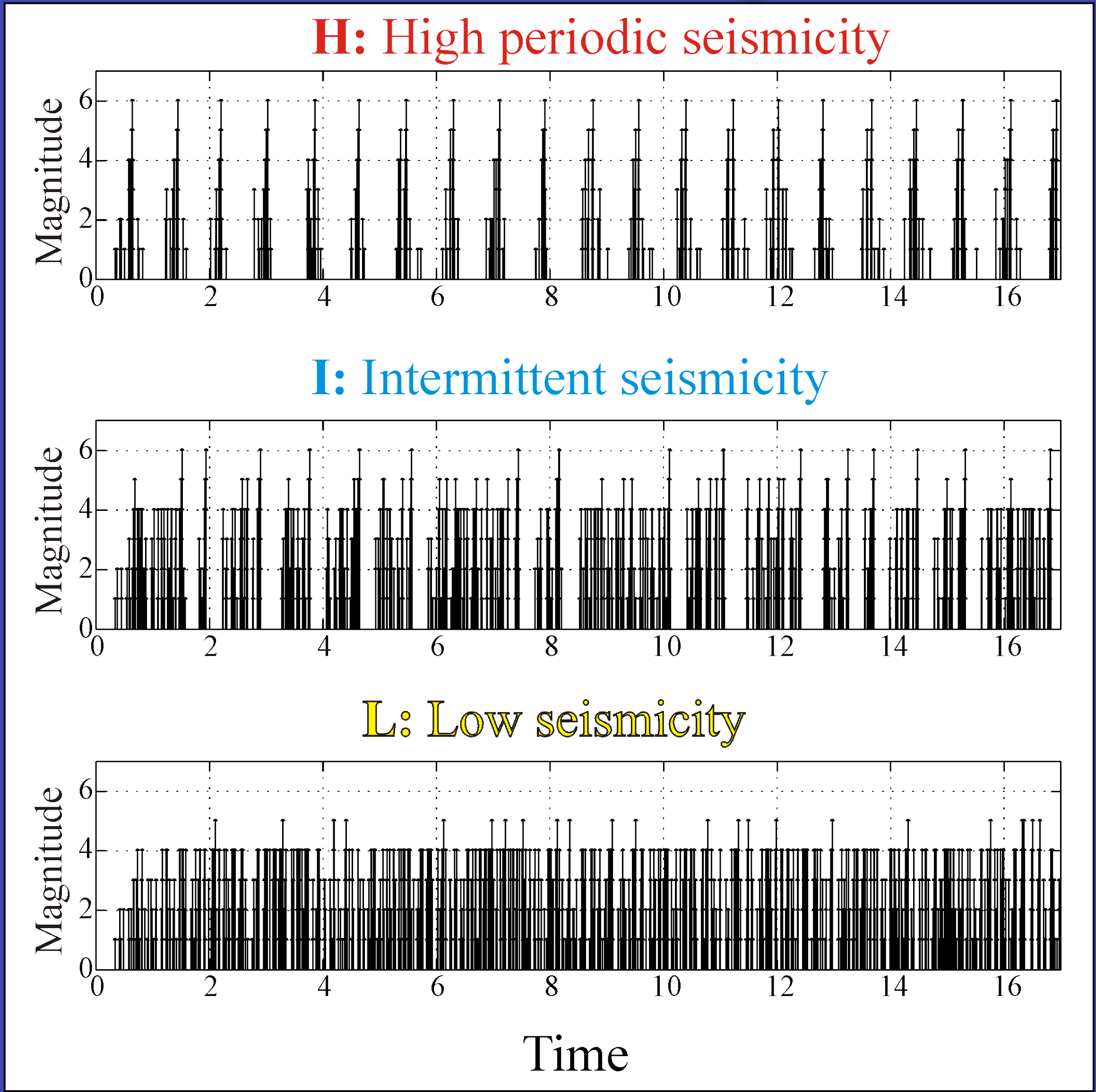


1. **Hierarchical structure**
2. **Loading by external forces**
3. **Elements' ability to fail & heal**



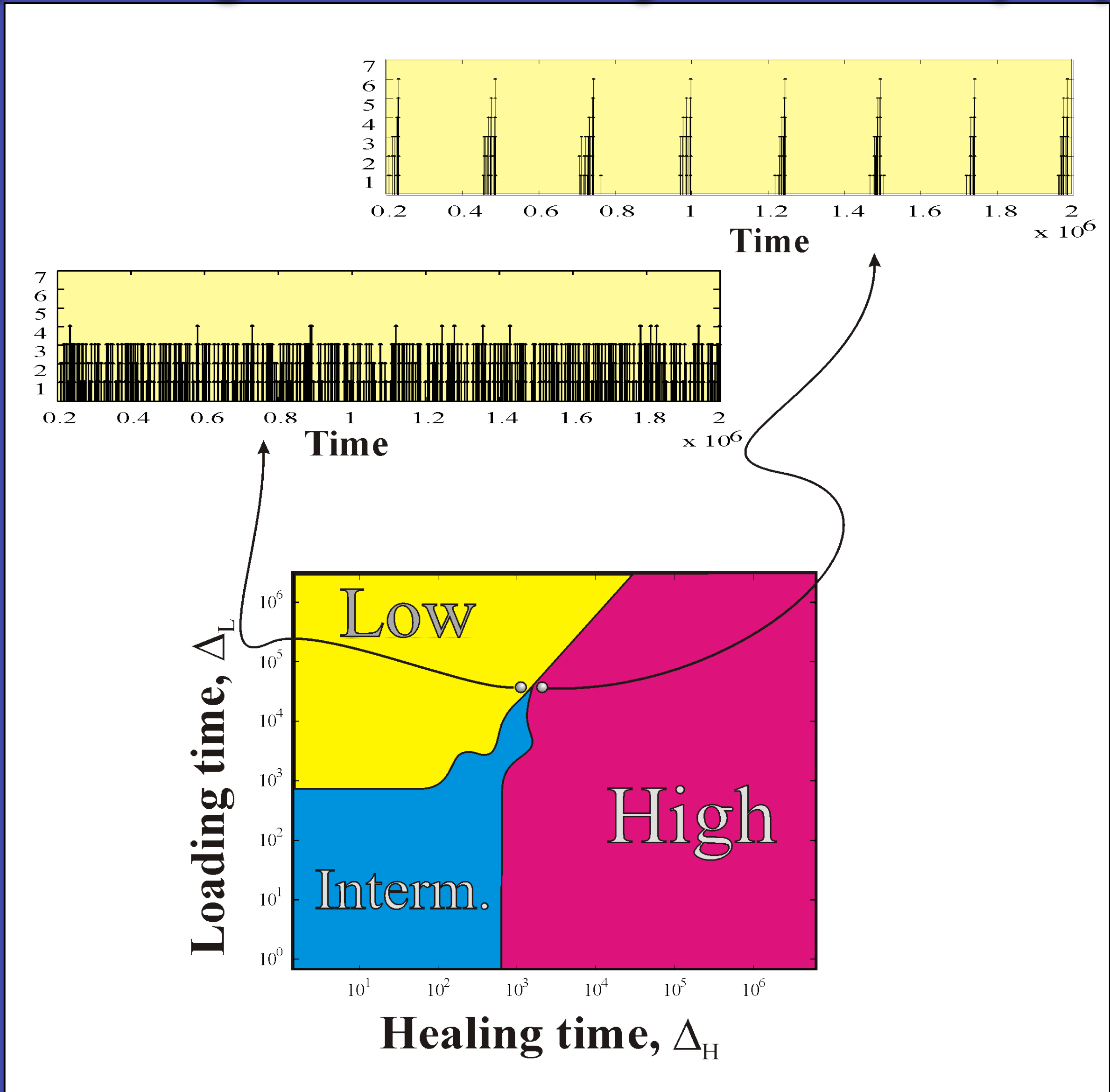
A. Gabrielov, V. Keilis-Borok, W. Newman, & I. Zaliapin (2000a, b, *Phys. Rev. E*; *Geophys. J. Int.*)

BDE model of colliding cascades:
Three seismic regimes



I. Zaliapin, V. Keilis-Borok & M. Ghil (2003a, *J. Stat. Phys.*)

BDE model of colliding cascades
Regime diagram: Instability near the triple point



I. Zaliapin, V. Keilis-Borok & M. Ghil (2003a, *J. Stat. Phys.*)

"Partial" BDEs – An Introduction, I

Consider on-off sites $u_i(t)$ on a line and

$u_i(t) = u_{i-1}(t-\vartheta_t) \Delta u_i(t-\vartheta_t) \Delta u_{i+1}(t-\vartheta_t)$,

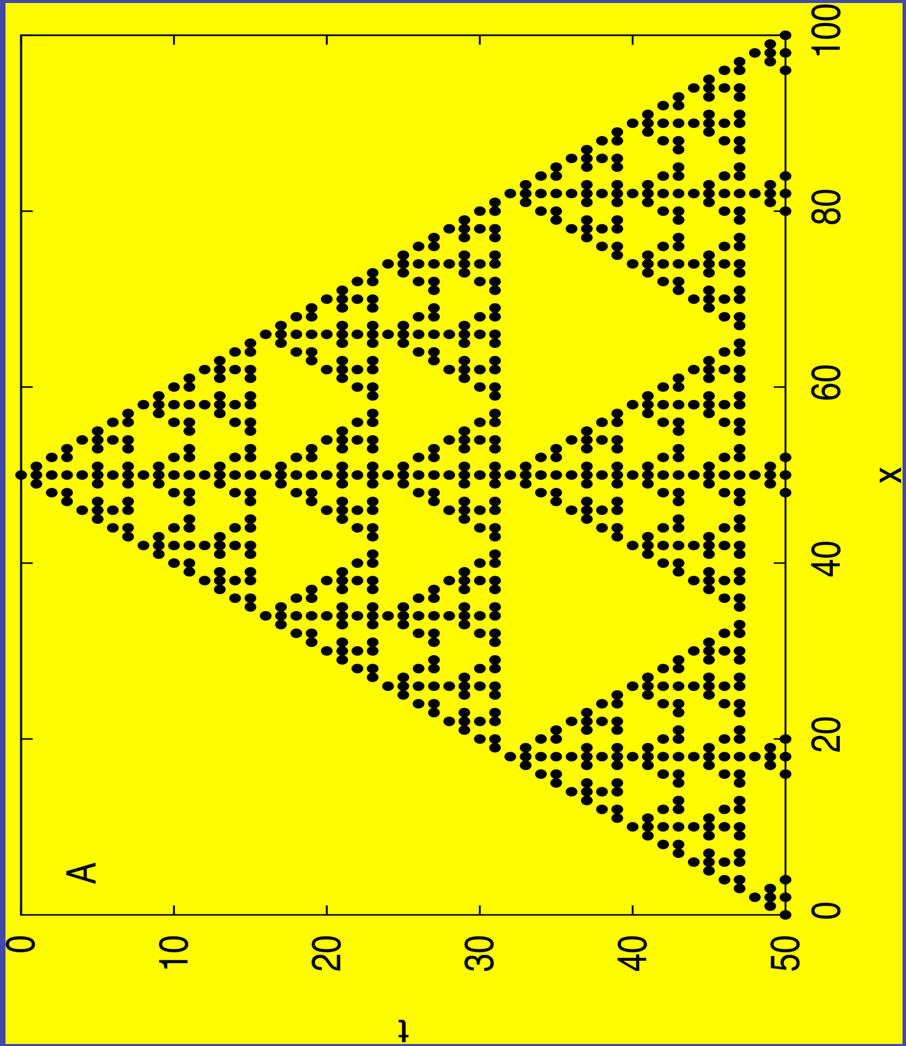
where Δ is the XOR operator, and
$\vartheta_t = const.$ for now is the time delay.

We use periodic boundary conditions,

$u_i(t) = u_{i+N}(t)$,

and thus have $n = 2N$ "ordinary" BDEs.

The initial state is $u_0(0) = 1$,
with all other $u_i(0) = 0$.



The evolution of the solution is the
"Pascal's triangle" in the figure.
For $\vartheta_t = const.$ it is equivalent to an elementary CA (ECA).

Ghil *et al.* (*Physica D*, 2008)

“Partial” BDEs – An Introduction, II

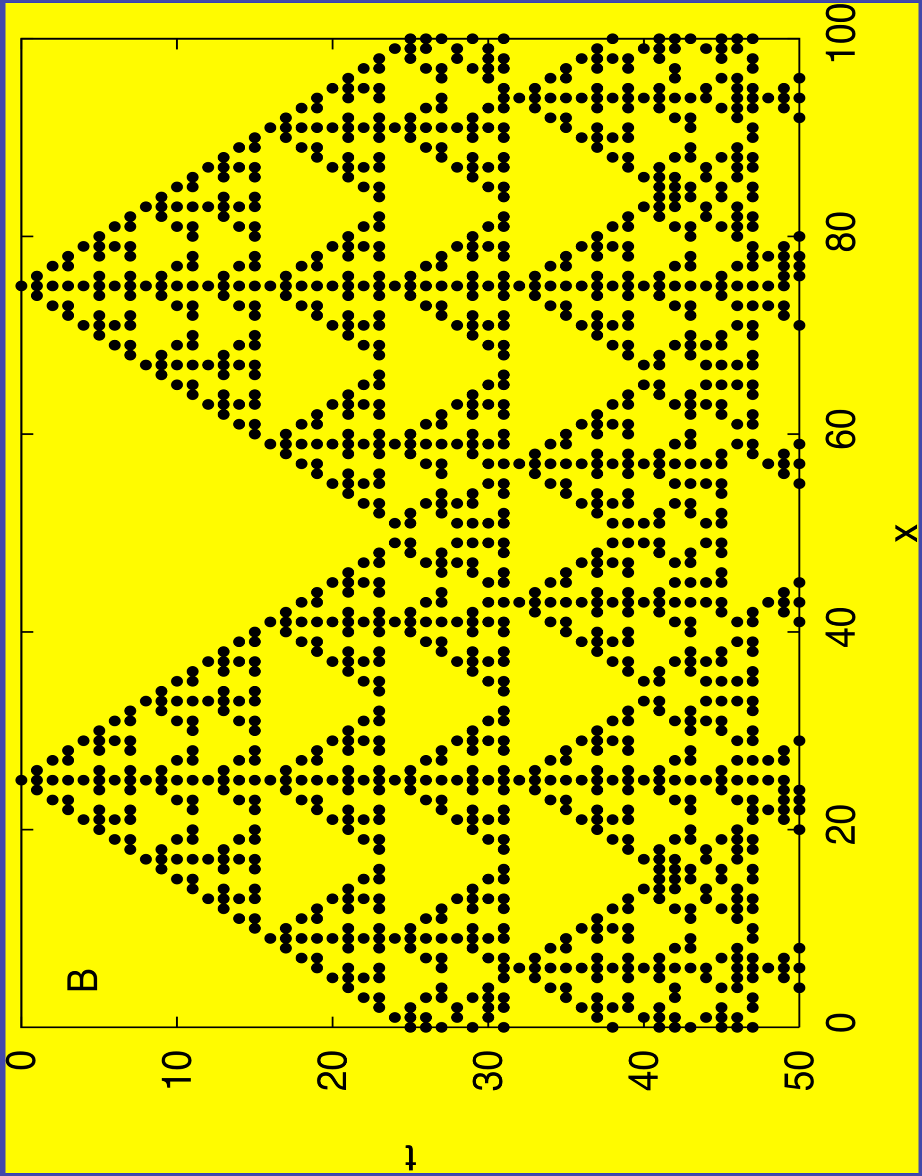
The figure now shows the “collision” of two waves, each started from an “on” site, while all other sites are “off.”

Thus the solution in the previous slide is a “Green’s function” of the partial BDE (PBDE) before.

This behavior is still equivalent to that of an ECA, as long as $\vartheta_t = const.$

But more interesting things will happen when that is no longer the case.

Empty sites, $u_i(t) = 0$ in white, while occupied sites, $u_i(t) = 0$ are in black.



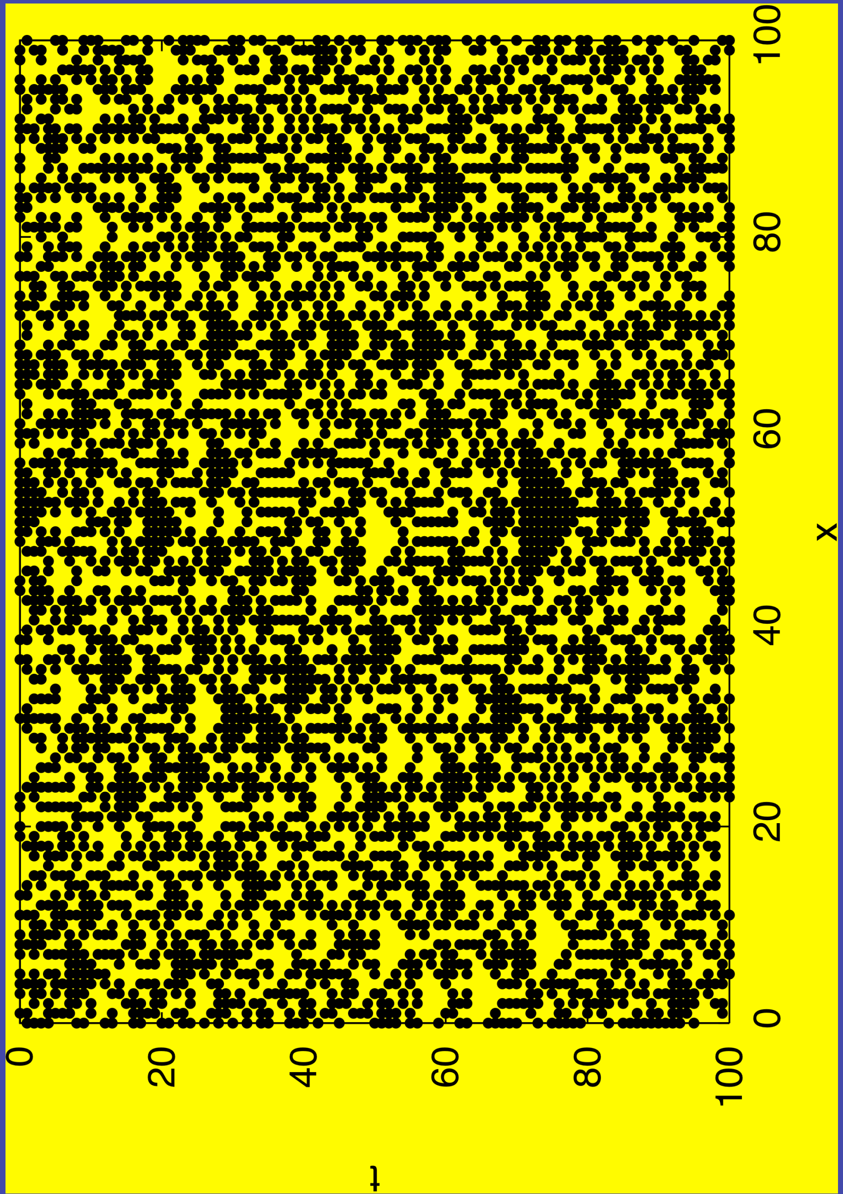
Ghil *et al.* (*Physica D*, 2008)

“Partial” BDEs – An Introduction, III

The figure now shows the solution of the same PBDE, when the initial state is a random distribution of “on” and “off” sites.

The qualitative behavior is characterized by “triangles" of empty (white) or occupied (black) sites, without any recurrent pattern.

This behavior does not depend on the particular random initial state.



Ghil *et al.* (*Physica D,* 2008)

Random Dynamical Systems - RDS theory

This theory is a combination of measure (probability) theory and dynamical systems developed by the “Bremen group" (L.Arnold, 1998). It allows one to treat Stochastic Differential Equations (**SDEs**), and more general systems driven by some “noise," as **flows**.

Setting:

(i) A phase space X. **Example**: $\mathbb{R}^n$.

(ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example**: The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P} = \gamma$.

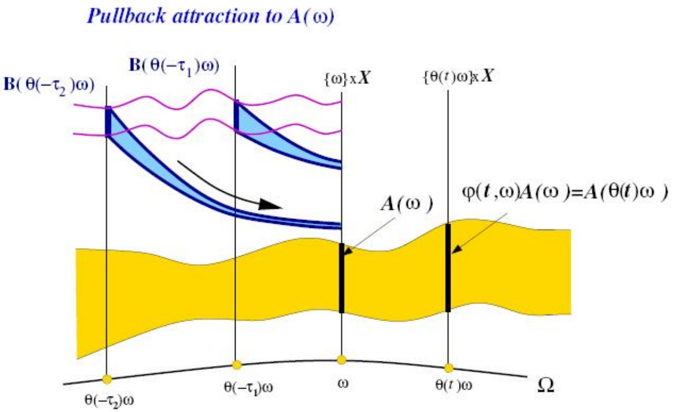
(iii) A model of the noise $\theta(t) : \Omega \to \Omega$ that preserves the measure $\mathbb{P}$, i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; θ is called the driving system. **Example**: $W(t, \theta(s)\omega) = W(t+s, \omega) - W(s, \omega)$; it starts the noise at s instead of $t = 0$.

(iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \to X$ with the cocycle property. **Example**: The solution of an SDE.

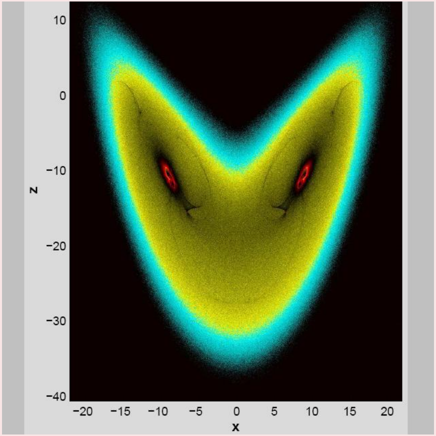
A random attractor $\mathcal{A}(\omega)$ is both *invariant* and "pullback" *attracting*:

(a) **Invariant**: $\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$.

(b) **Attracting**: $\forall B \subset X$, $\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$ a.s.

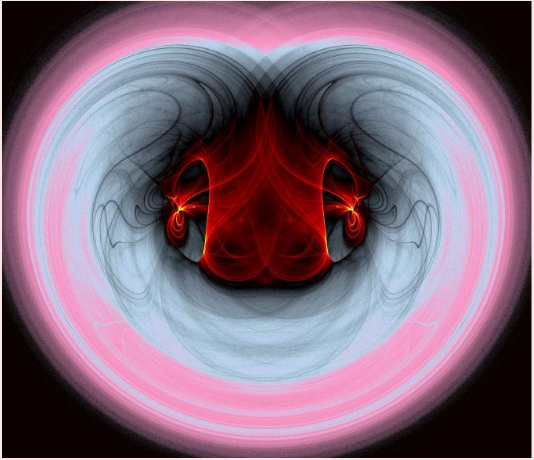


Disintegration of the measure supported by the Lorenz R.A.

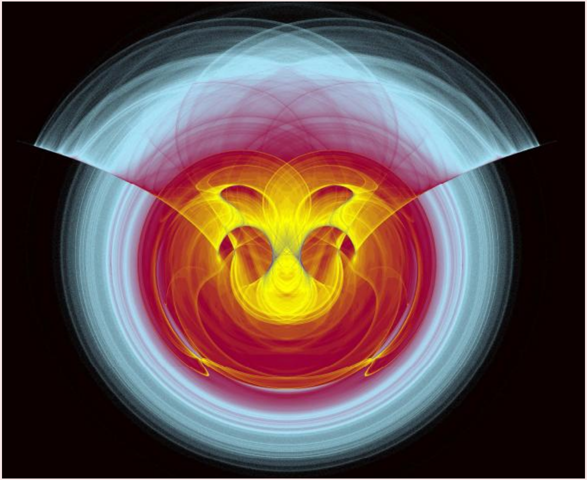


- We can compute the probability measure on the R.A. at some fixed time t. We show a "projection", $\int \mu_\omega(x, y, z)\mathrm{d}y$, with multiplicative noise: $\mathrm{d}x_i$=Lorenz$(x_1, x_2, x_3)\mathrm{d}t$ + α $x_i \mathrm{d}W_t$; $i \in \{1, 2, 3\}$.
- 10 million of initial points have been used for this picture!

Disintegration of the measure supported by the R.A.

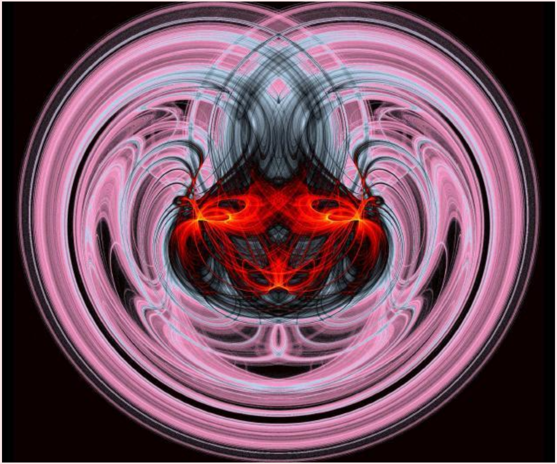


- Still 1 Billion I.D., and $\alpha = 0.3$.



- Here $\alpha = 0.4$. The sample measure is approximated for another realization of the noise, starting from 8 billion I.D.
- Now more serious stuff is coming...

Disintegration of the measure supported by the R.A.



- Still 1 Billion I.D., and $\alpha = 0.5$. Another one?

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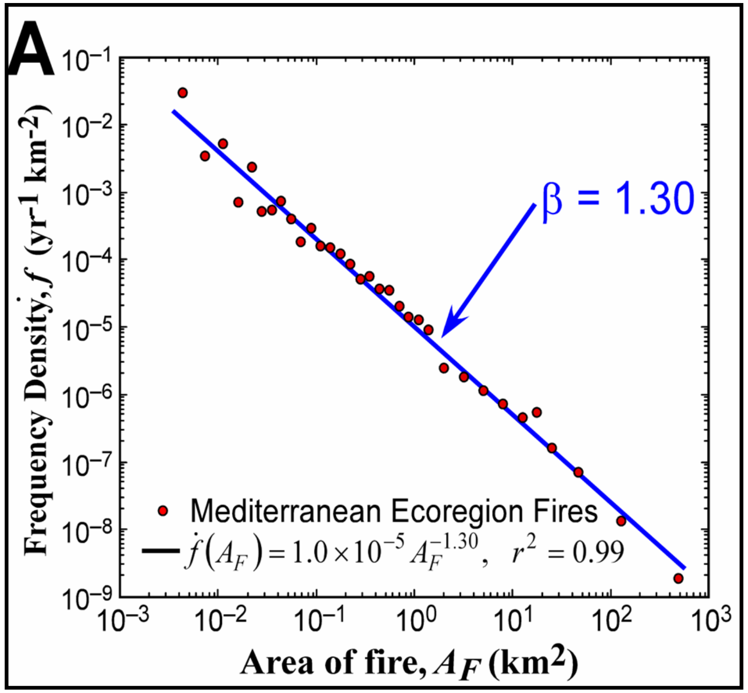
Concluding remarks – I

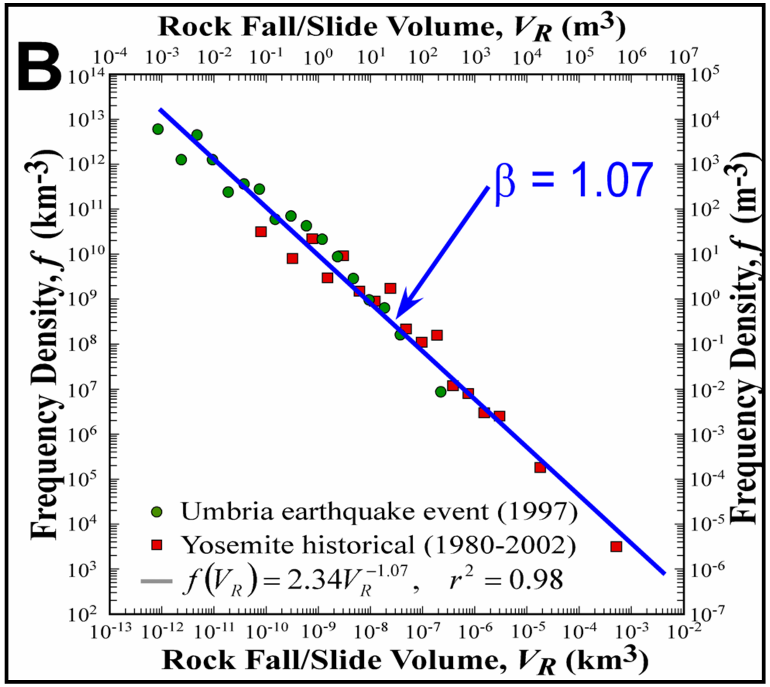
Fractals are fun and they are quite useful, too.

USA Wildfires



Malamud, Morein & Turcotte (1998, *Science*)

USA & Italian Rockfalls



Malamud, Turcotte et al. (2004, *ESPL*)

Frequency-size distributions for natural hazards

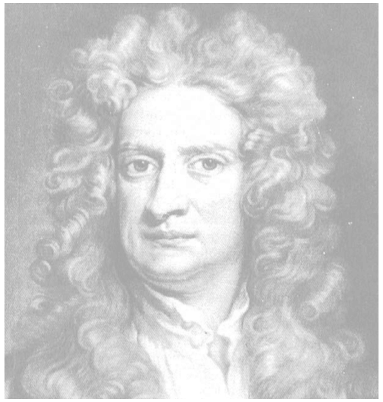
➔ probabilistic hazard forecasting

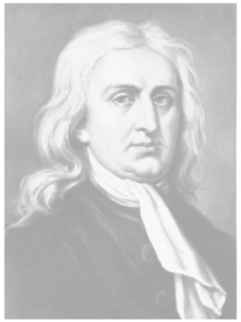
Concluding remarks – II

Fractals are fun and they are quite useful, too.

Benoît was the Adam and the Kepler of fractals.

We still need a few Newtons …





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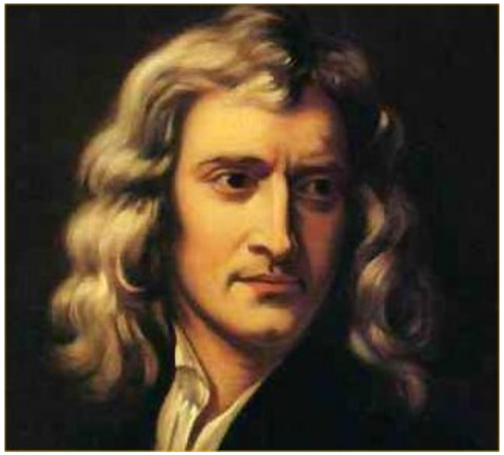

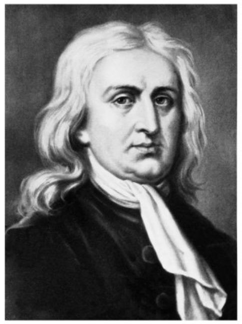

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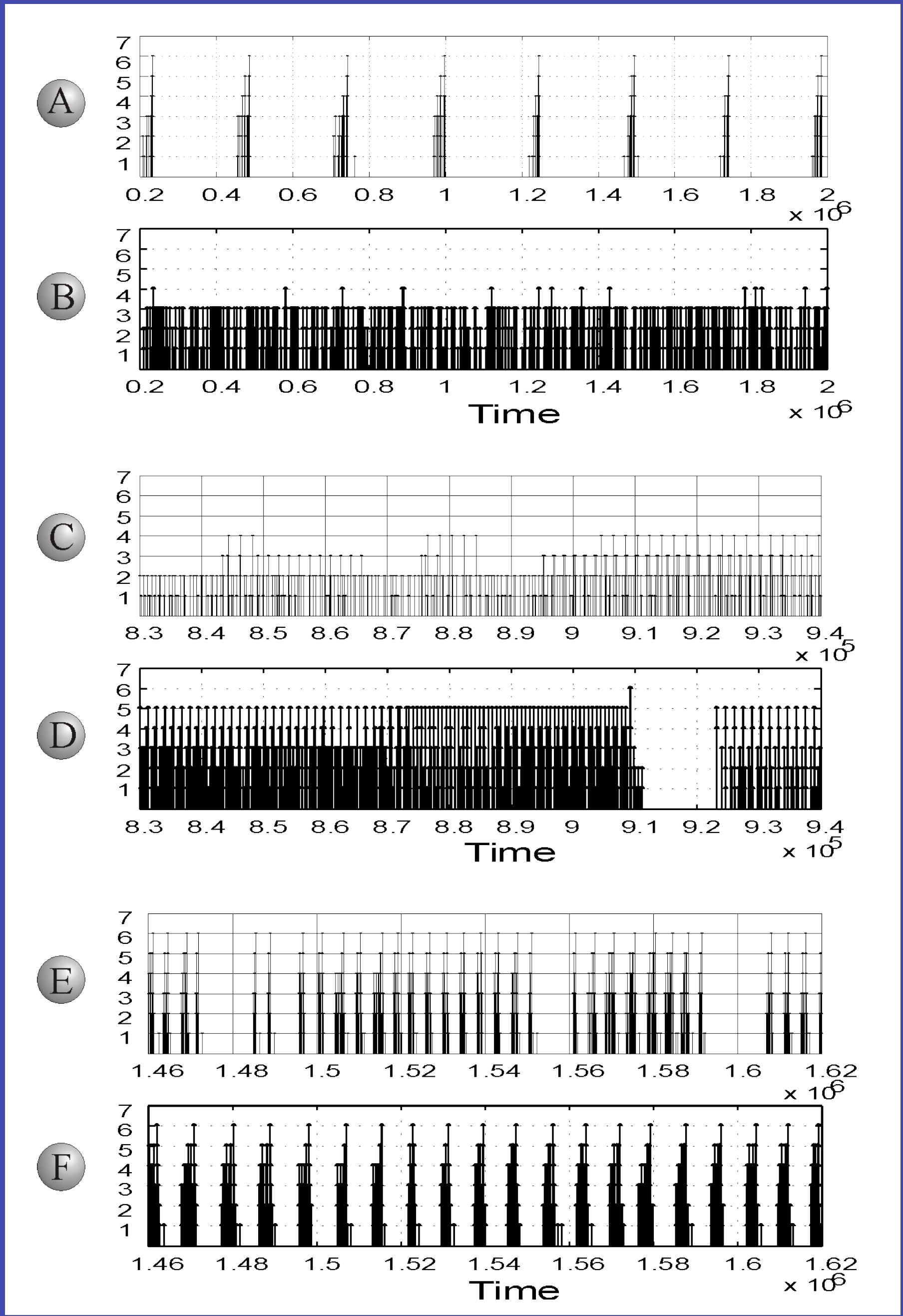
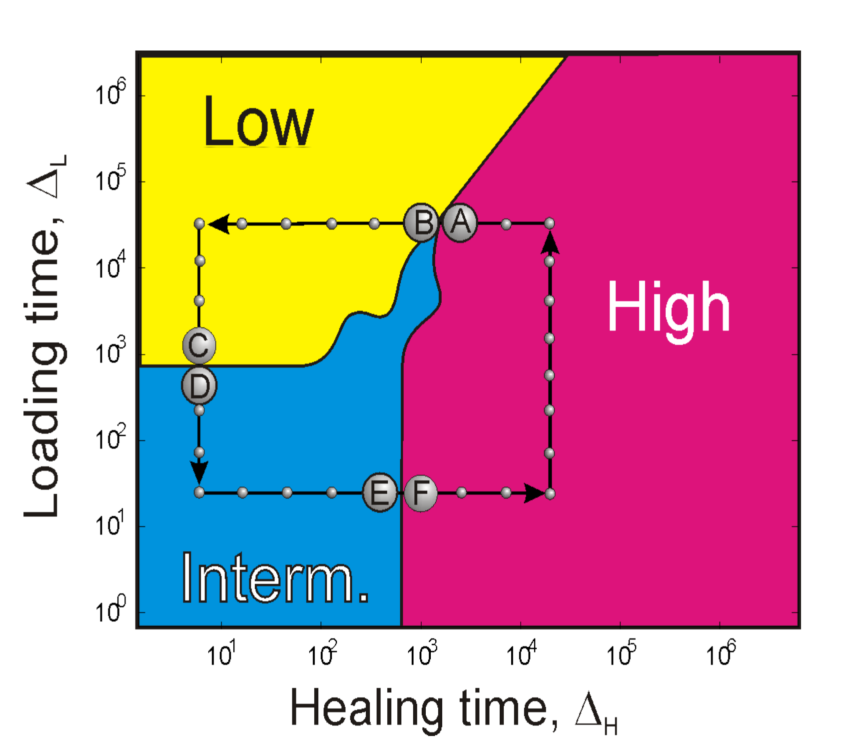
Mandelbrot,B., 1982: *The Fractal Geometry of Nature*, W. H. Freeman & Co, San Francisco.

Peitgen, H.-O., and P. Richter, 1986: *The Beauty of Fractals.* Springer-Verlag. Heidelberg.

Reserve slides

BDE model of colliding cascades
Regime diagram:
Transition between regimes





I. Zaliapin, V. Keilis-Borok & M. Ghil
(2003a, *J. Stat. Phys.*)

Extreme Events:
Causes and Consequences (E2-C2)

- EC-funded project bringing together researchers in mathematics, physics, environmental and socio-economic sciences.
- €1.5M over 3.5 years (March 2005–August 2008).
- Coordinating institute: Ecole Normale Supérieure.
- 17 ‘partners’ in 9 countries.
- 72 scientists + 17 postdocs/postgrads.
- PEB: M. Ghil (ENS, Paris, P.I.), S. Hallegatte (CIRED), B. Malamud (KCL, London), A. Soloviev (MITPAN, Moscow), P. Yiou (LSCE, Gif s/Yvette, Co-P.I.)