From *SIAM News*, Volume 44, Number 9, November 2011

ICIAM 2011

Toward a Mathematical Theory of Climate Sensitivity

By Michael Ghil

The first attempt at a consensus estimate of the equilibrium sensitivity of climate to changes in atmospheric carbon dioxide concentrations appeared in 1979, in the U.S. National Research Council report of J.G. Charney and associates. The result was the now famous range for an increase of 1.5–4.5 K in global temperatures, given a doubling of CO_2 concentrations.

Earth's climate, however, never was and is unlikely ever to be in equilibrium. The Intergovernmental Panel on Climate Change, therefore, in addition to estimates of equilibrium sensitivity, focused on estimates of climate change in the 21st century. The latter estimates of temperature increase in the coming 100 years still range over several degrees Celsius. This difficulty in narrowing the range of estimates is clearly connected to the complexity of the climate system, the nonlinearity of the processes involved, and the obstacles to a faithful representation of these processes and feedbacks in global climate models, as described in [4].

My talk at ICIAM 2011 reflected joint work with Mickaël D. Chekroun and Dmitri Kondrashov (UCLA), Eric Simonnet (Institut Non Linéaire de Nice), Shouhong Wang (Indiana University), and Ilya Zaliapin (University of Nevada, Reno). The main objective of our work is to understand and explain, at a fundamental level, the causes and manifestations of climate sensitivity. This work is based on a weaving together of recent results from three mathematical disciplines: the ergodic theory of dynamical systems, stochastic processes, and the response theory of nonequilibrium dynamical systems. The cornerstone is the theory of *random dynamical systems*, which allows us to probe the detailed geometric structure of the random attractors associated with nonlinear, stochastically perturbed systems. These attractors extend the concept of strange attractors from autonomous dynamical systems to non-autonomous and stochastic systems.

In fact, the theory of differentiable dynamical systems—as we know and love it from the work of G.D. Birkhoff, J. Hadamard, H. Poincaré, and, more recently, E.N. Lorenz, D. Ruelle, and S. Smale, among many others—applies to autonomous systems, in which neither the forcing nor the coefficients depend explicitly on time. This theory is well suited for the study of physical, chemical, biological, or social systems that are closed, i.e., can be completely isolated from their surroundings. Such is certainly not the case of the earth's climate system, which receives energy from the sun and returns it to interplanetary space. Moreover, depending on the time scale of interest, one often wishes to study only part of the climate system. Thus, in numerical weather prediction out to a mere few days, one tends to neglect the intrinsic variability of the oceans and concentrates on the atmosphere, with sea surface temperatures prescribed as a boundary condition; the sea surface temperature field can either be kept constant in time or allowed to vary in some prescribed manner, e.g., according to a diurnal cycle. The same can be said about various coefficients that enter the atmosphere's governing partial differential equations.

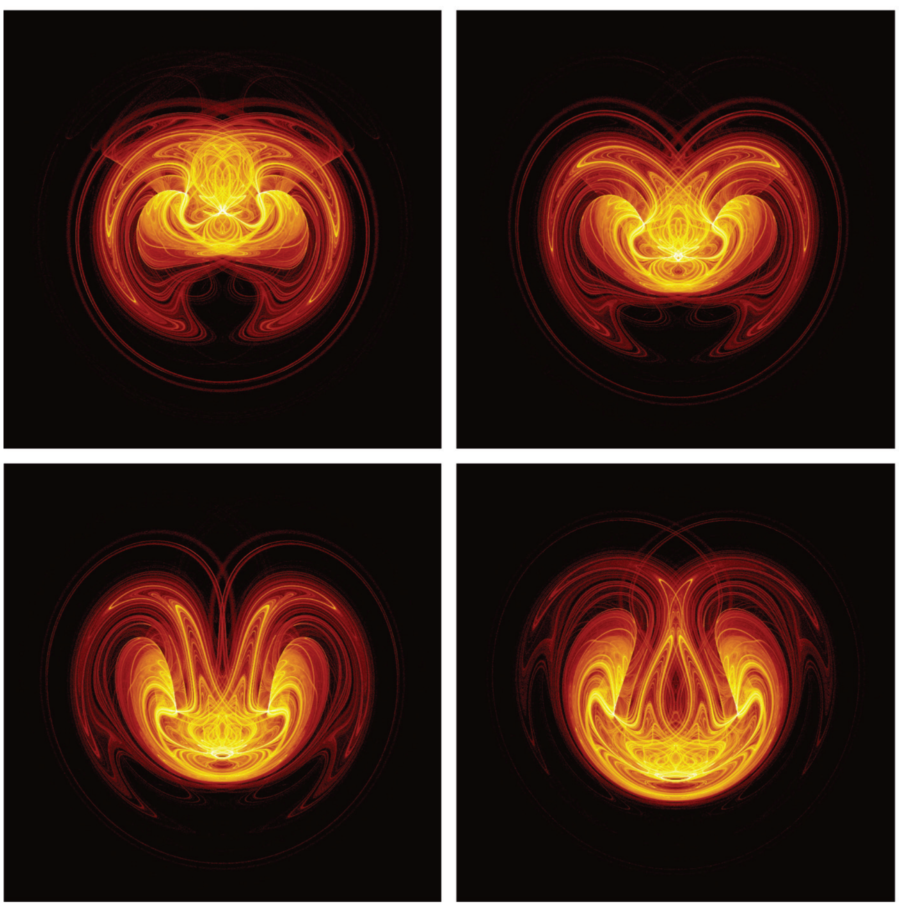

Figure 1. *Snapshots of the random attractor of a classic Lorenz model.*

The theoretical underpinnings of the study of the dynamical behavior of open systems—which are in contact with their surroundings and thus may exhibit time dependence in their forcing or coefficients—were laid within the last couple of decades by L. Arnold, G. Sell, and L.-S. Young, among many others. In the presence of dissipation, one still expects convergence of the phase-space flow to some lower-dimensional object. But this object, termed a *pullback attractor* in the deterministic context and a *random attractor* in the stochastic one, is now itself time dependent. To see this attractor at time t, we need to pull back to a time $s \ll t$ and let the phase space flow onto the attractor. The theory requires $s \to -\infty$, but the numerical practice shows that the requisite duration $t-s$ of "pulling back" depends on the system's degree of dissipativity and can often be fairly short.

At ICIAM, to illustrate our results so far, I described high-resolution numerical studies of several "toy" models, for which we obtained good approximations of their global pullback or random attractors, as well as of the time-dependent invariant measures supported by these attractors. The latter measures were shown to be random Sinai–Ruelle–Bowen measures; it is these measures that have an intuitive, physical interpretation: They are obtained essentially by "flowing" the entire phase space onto the attractor.

The first of the models we studied is a stochastically forced version of the classic Lorenz model (1963). Several snapshots of its random attractor are shown in Figure 1; a short video clip of the attractor's evolution in time can be found in the supplementary material of Chekroun et al. [1]. The second one is a low-dimensional, nonlinear stochastic model of the El Niño–Southern Oscillation; Figure 2 shows successive snapshots of its random attractor, over a full ENSO cycle. While highly idealized, both these models are of fundamental interest for climate dynamics and provide insight into its predictability. More on the predictability of a randomly driven ENSO model and of ENSO itself can be found in [2].

Finally, I provided an outlook on response theory as applied to random dynamical systems, rather than in the more familiar context of statistical mechanics near equilibrium. This theory provides the response function $R(t)$ of a chaotic system to time-dependent forcing, as well as its Fourier transform, the susceptibility function (ξ). In fact, climate change involves not just changes in the mean, but also in its variability [3]. Thus, the susceptibility function will allow us to get a handle on mechanisms of high sensitivity in the response of climate variability to deterministic, anthropogenic forcing—such as increases in aerosols and greenhouse gases—as well as to random, natural forcing, such as volcanic eruptions.

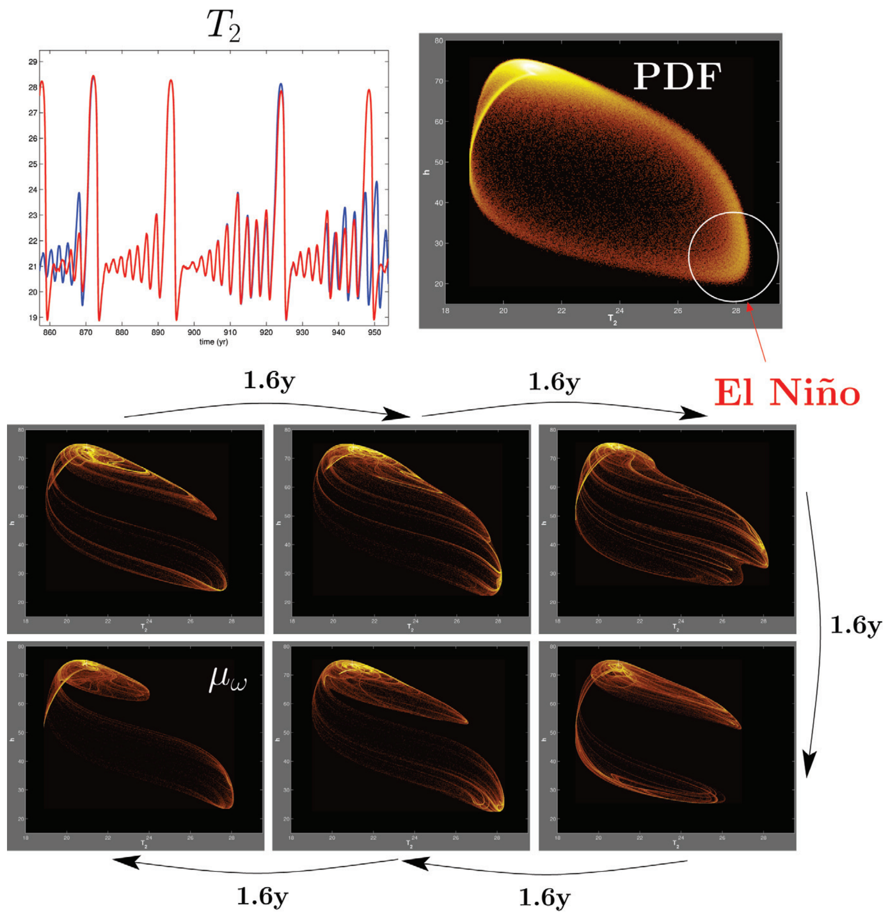

Figure 2. *Successive snapshots of the random attractor of the El Niño–Southern Oscillation model.*

References

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