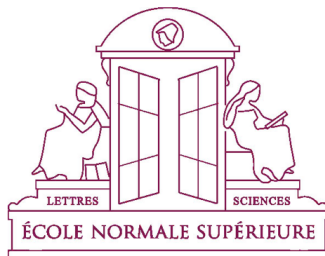


# ***What is a Tipping Point & Why Do We Care?***

**Michael Ghil**

**Ecole Normale Supérieure, Paris, and  
University of California, Los Angeles**




***Please visit these sites for more info.***

<http://www.atmos.ucla.edu/tcd/>

<http://www.environnement.ens.fr/>

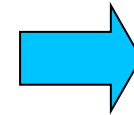
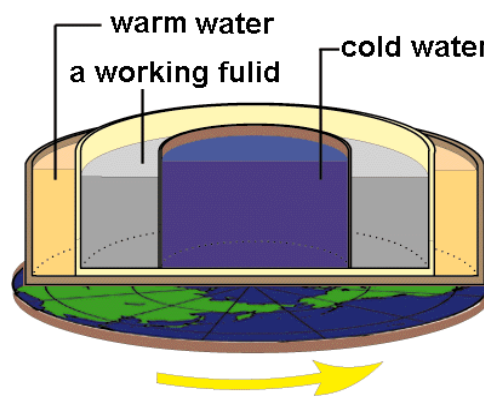
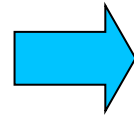
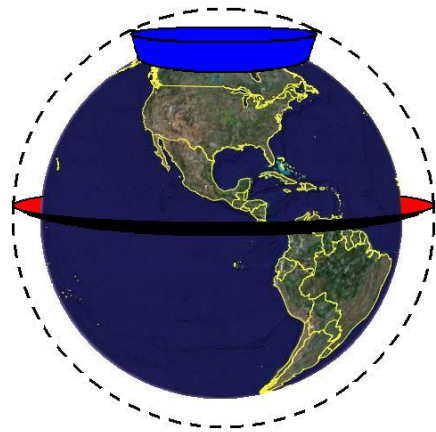
# Motivation

- There's a lot of talk about "tipping points."
- It sounds threatening, like falling off a cliff: that's why we care!
- But what are they, and what do we know about them?
- Here's a disambiguation page (cf. Wikipedia), first.
- **Sociology**: "the moment of critical mass, the threshold, the boiling point" (Gladwell, 2000); a previously rare phenomenon becomes rapidly and dramatically more common.
- **Physics**: the point at which a system changes from a stable equilibrium into a new, qualitatively dissimilar equilibrium (throwing a switch, tilting a plank, boiling water, etc.).
- **Climatology**: "A climate tipping point is a somewhat ill-defined concept [...]" — so we'll try to actually define it better.  

- **Catastrophe theory**: branch of bifurcation theory in the study of dynamical systems; here, a tipping point is "a parameter value at which the set of equilibria abruptly change." → **Let's see!**

M. Gladwell (2000) *The Tipping Point: How Little Things Can Make a Big Difference*.

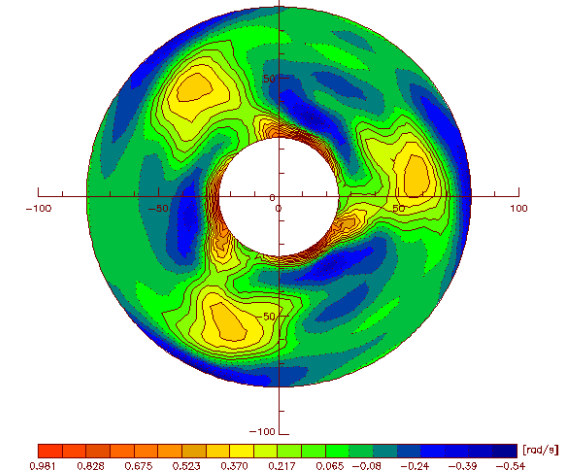
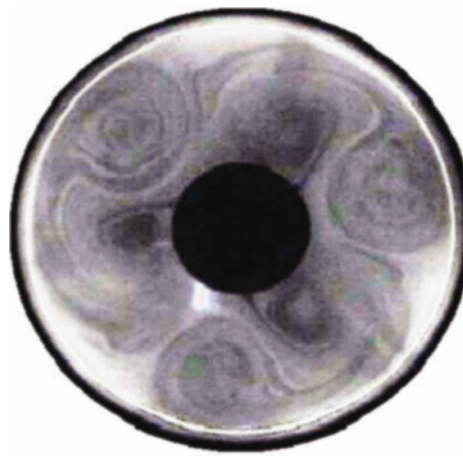
T. M. Lenton *et al.* (2008) Tipping elements in the Earth's climate system, *PNAS*, v. **105**.

# Laboratory Analogues of Planetary Atmospheric Circulation Systems



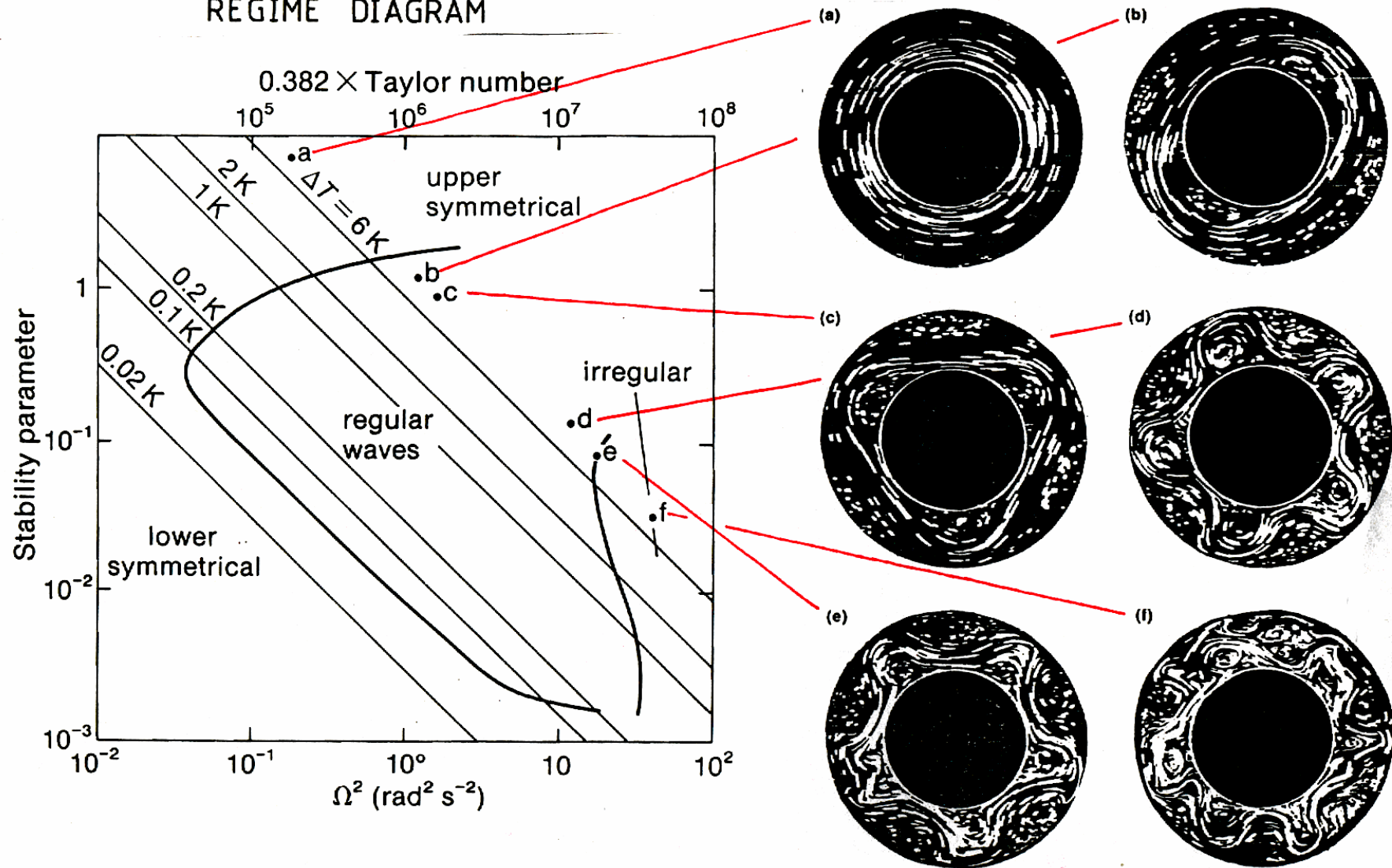
Const.  $\Delta T = 4.01^\circ\text{C}$  Level 1  
Relative vorticity (vert. comp.) exp1/101  
Time Interval: 2000-2025 s

- *Baroclinic instability*  
- a potential energy releasing instability in the atmosphere and oceans



# Rotating Convection: An Illustration

TYPICAL FLOWS AND  
REGIME DIAGRAM



# *Outline, Tipping Points I*

## **Elementary Bifurcation Theory and Variational Principle**

- **Fixed Points**
    - linear stability
    - nonlinear stability and attractor bassins
  - **Saddle-node bifurcations**
    - multiple branches of stationary solutions
    - linear stability
  - **Bifurcations in 1-D**
  - **Nonlinear stability and variational principle**
    - variational principle in 0-D
    - variational principle in 1-D
5. **Bistability and hysteresis**

# 1. Points fixes

Nous avons étudié les solutions des *EBMs* en *0-D* :

$$\dot{T} = f(T; \mu, c, \sigma, \dots)$$

La forme générale du problème scalaire s'écrit :

$$\dot{x} = f(x; \mu)$$

*Stabilité* — linéaire,  $\mu = 1$

$$f(x_0) = 0 \quad \xrightarrow{\quad x_0 \quad} \quad x \quad (2)$$

$$\dot{x} = 0 \Rightarrow x \equiv x_0 - \textit{point fixe}$$

Perturbation initiale,  $t = 0$

$$\begin{aligned} x(0) &= x_0 + \xi(0) \\ \dot{x} &\equiv \dot{x}_0 + \dot{\xi} = \dot{\xi} \\ &= f(x_0 + \xi) = f(x_0) + f'(x_0)\xi + O(\xi^2) \end{aligned}$$

*Stabilité linéaire* (~ « perturbations infinitésimales »)

$$\dot{\xi} = f'(x_0)\xi, \quad f'(x_0) \equiv \lambda, \quad \dot{\xi} = \lambda\xi \quad (3)$$

$$\xi(t) = e^{\lambda t} \xi_0, \quad \xi_0 \equiv \xi(0)$$

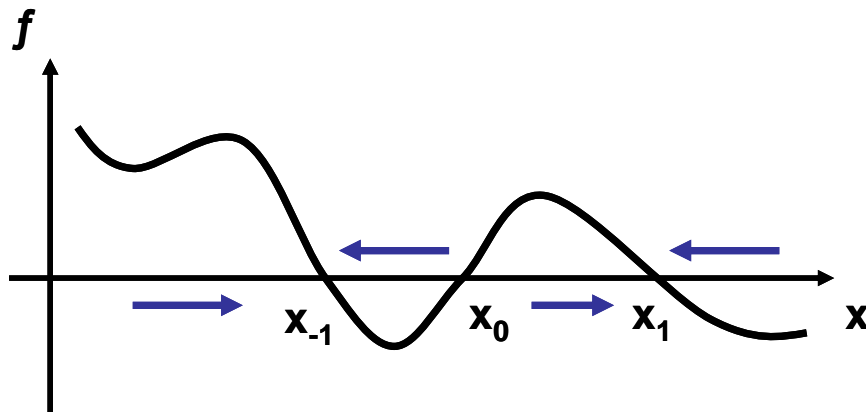
Si  $\lambda < 0 \Rightarrow$  le point fixe (PF) est (linéairement) *stable*

Si  $\lambda > 0 \Rightarrow$  le PF est " *instable*

Si  $\lambda = 0 \Rightarrow$  la stabilité linéaire du PF est *neutre*

### Quelques généralités sur les PFs

- 1)  $f \in C^1, f \neq 0$  sur tout sous-intervalle : les PFs sont isolés (propriété générique)
- 2) Les bassins d'attraction sont des intervalles ouverts (éventuellement semi-infinis)



## 2. Bifurcation de col

Comment change-t-elle la géométrie des solutions quand  $\mu \neq \mu_0$ , ç.a.d. comment changent le nombre et la stabilité des solutions stationnaires ? Nous commençons par l'étude du cas scalaire.

Le cas le plus simple : col ou point-limite

$$\dot{x} = \mu - x^2 \equiv f(x; \mu) \quad (4)$$

$$\text{PFs: } \mu - x^2 = 0$$

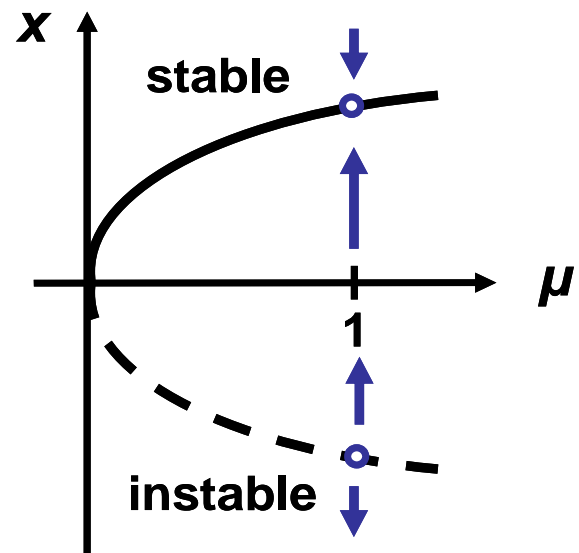
$$x = \pm\sqrt{\mu} \quad (5)$$

Stabilité des PFs :

$$x_1 = \sqrt{\mu}, \quad x_{-1} = -\sqrt{\mu}$$

$$x(0) = x_{\pm 1} + \xi(0)$$

$$\begin{aligned} \dot{\xi} &= \lambda_{\pm} \xi, & \lambda_{\pm} &\equiv f'(x_{\pm 1}) = -2x_{\pm 1} \\ & & &= \mp 2\sqrt{\mu} \end{aligned} \quad (6)$$



∴ Branche supérieure – (linéairement) *stable*

Branche inférieure – (linéairement) *instable*

Nous passons maintenant à l'étude de la stabilité non-linéaire.



### 3. Bifurcations en $n$ -D

Nous avons étudié le cas scalaire  $n = 1$ . De façon plus générale, on aura :

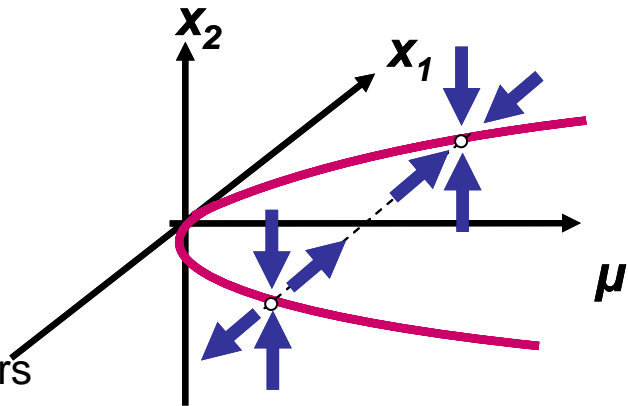
$$\dot{\vec{x}} = \vec{f}(\vec{x}; \mu) \quad , \quad \vec{f} \in C(\mathbb{R}^n \times \mathbb{R})$$

avec  $\vec{x} \in \mathbb{R}^n$  et  $\mu \in \mathbb{R}$ .

Le comportement est “presque” linéaire partout dans l’espace de phase-paramètre  $\mathbb{R}^n \times \mathbb{R}$ , sauf dans le voisinage de quelques points isolés  $(x_c, \mu_c)$  : c’est les points de bifurcation, où la matrice jacobienne  $L = (\partial f_i / \partial x_j)$  est singulière, c’est-à-dire  $\det L = 0$ .

Dans le cas  $n = 2$ , on peut réduire à la *forme normale* :

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2, \\ \dot{x}_2 &= -\lambda x_2, \lambda > 0. \end{aligned}$$



Ce schéma explique d’ailleurs l’appellation “bifurcation de col”.

Dans le cas général, la réduction donne :

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2 \\ \dot{x}_i &= -\lambda_i x_i, \lambda_i > 0, i = 2, \dots, n. \end{aligned}$$

## 4. Stabilité non-linéaire et principe variationnel

Pour approfondir nos connaissances sur la stabilité, il faut étudier l'effet des perturbations plus importantes.

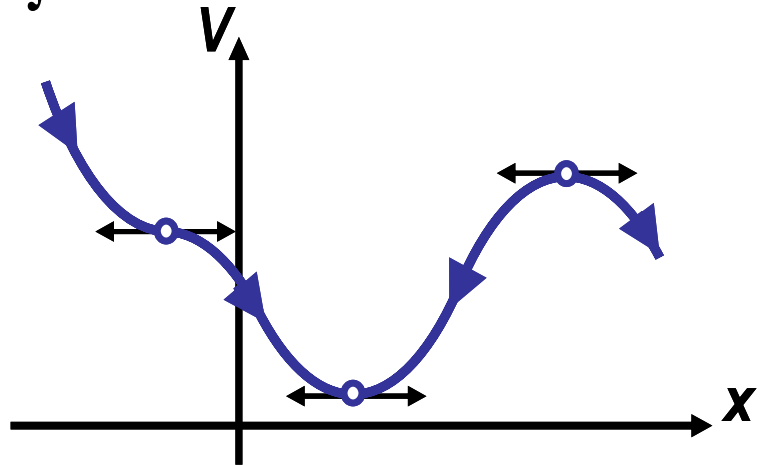
### a) Principe variationnel en 0-D

$$\begin{aligned} \dot{x} &= f(x) & V(x) &= -\int^x f(\xi) d\xi \text{ -- pseudo-potentiel} \\ &= -V'(x) \\ \dot{x}^2 &= -\frac{dV}{dx} \frac{dx}{dt} = -\dot{V} \end{aligned}$$

$\therefore V$  va décroître le long de la trajectoire de l'EDO tant que

$$\dot{x} \neq 0 \Leftrightarrow V' \neq 0$$

$\dot{x} = 0$  si  $V$  atteint un minimum, maximum ou col.

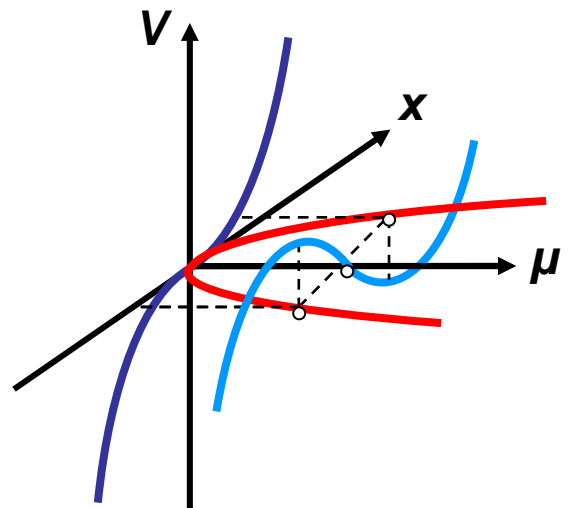


Evidemment, seulement  $V = \mathbf{min}$  est *stable* – de façon non-linéaire.

Avec ce résultat, nous retournons à la bifurcation de col.

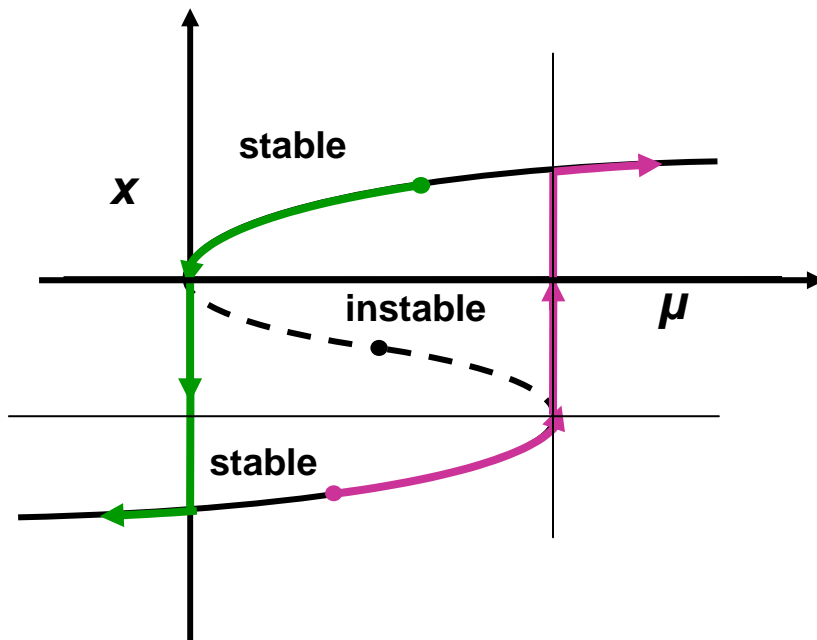
$$\dot{x} = \mu - x^2,$$

$$V(x; \mu) = -\mu x + x^3 / 3 + c(\mu).$$



## 5. Bistabilité et hystérésis

L'association de deux bifurcations de col peut créer un phénomène d'hystérésis (courbe en forme de S) :



$\dot{x} = \mu - x^2$  : la bifurcation en haut à gauche

$\dot{x} = (\mu - 1) + (x + \frac{1}{2})^2$  : la bifurcation en bas à droite

# IMAGES SIMPLIFIEES DE LA CIRCULATION MOYENNE

## Circulation directe de Hadley

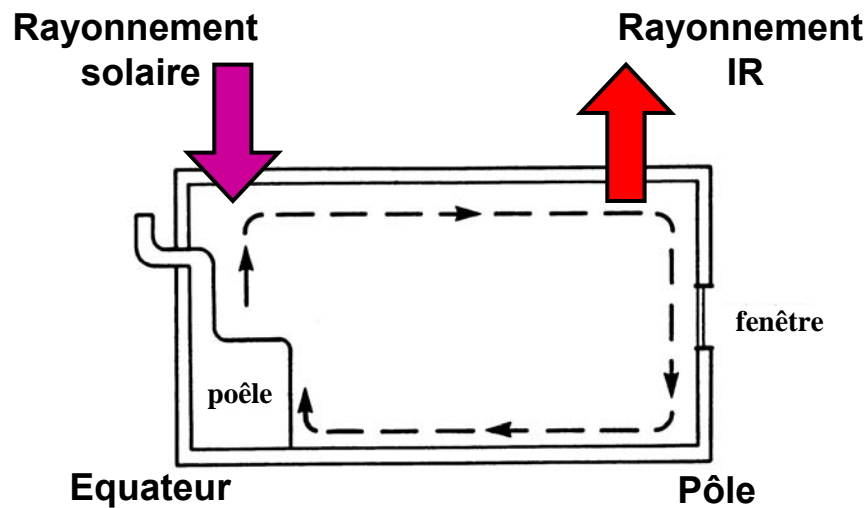


Image idéalisée de la circulation globale de l'atmosphère. \*

## Circulation observée

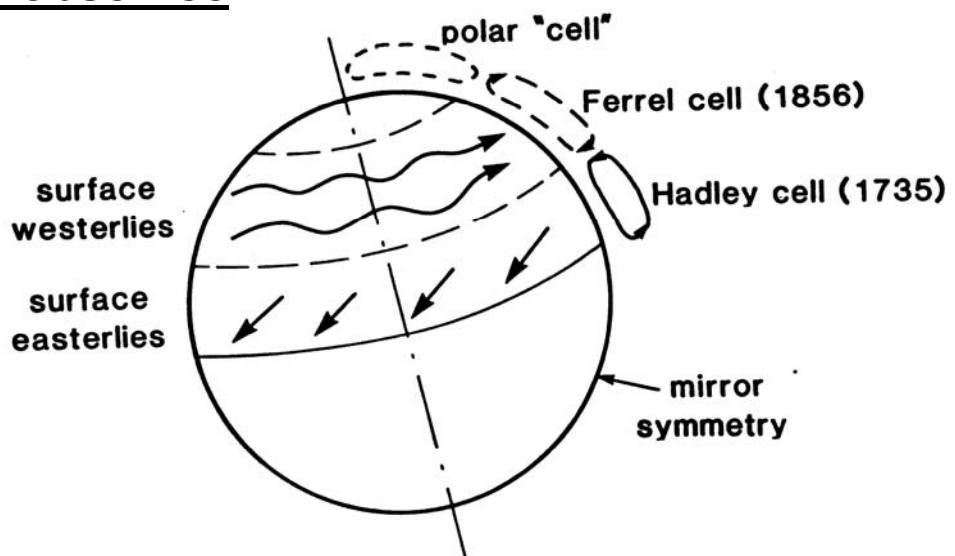


Schéma de la circulation générale de l'atmosphère. \*

---

\* D'après Ghil and Childress, Chap. 4

# **Modèles de bilan énergétique**

(Energy-balance models, EBMs)

$$C \frac{\partial T}{\partial t} = R_i - R_o + D$$

$C$  — capacité calorifique locale

$T$  — température de surface locale

$R_i$  — rayonnement solaire incident

$R_o$  — rayonnement terrestre vers l'espace

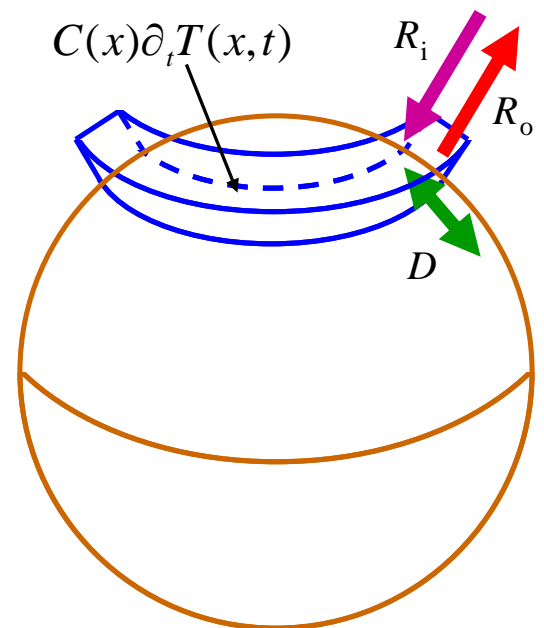
$D$  — redistribution de la chaleur (“diffusion”)

## **Remarques :**

1. une variable d'espace maximum ( $x$ )
2.  $C$ ,  $R_i$ ,  $R_o$  et  $D$  doivent être calculés (“paramétrés”) en fonction de  $T=T(x,t)$
3. Principale caractéristique du modèle :

$$R_i = Q(x) \{1 - \alpha(x, T)\}$$

avec  $\alpha$  l'albédo local



## ***Energy-balance models (EBMs)***

### **Version O-D (moyennée sur le globe)**

$$\begin{aligned} C \frac{d\bar{T}}{dt} &= R_i - R_o \\ &= Q \{1 - \alpha(\bar{T})\} - \sigma \bar{T}^4 m(\bar{T}) \end{aligned}$$

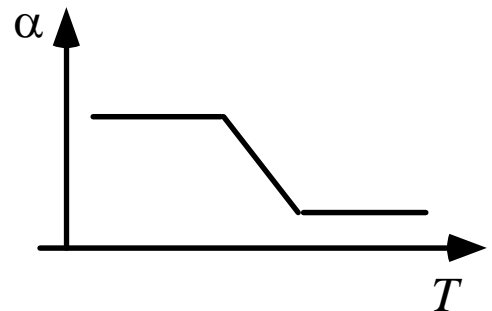
$\bar{T}$  — température moyenne à la surface

$t$  — temps (en milliers d'années)

$Q$  — flux solaire incident

$\alpha$  — albédo

$C$  — capacité calorifique



$\sigma$  — constante de Stefan–Boltzmann

$m$  — facteur d'effet de serre

Remarque:  $\alpha$  est fonction de l'extension de la glace et de la neige, de la nébulosité, etc. (variables implicites), tout étant paramétré en fonction de la variable explicite  $\bar{T}$ .

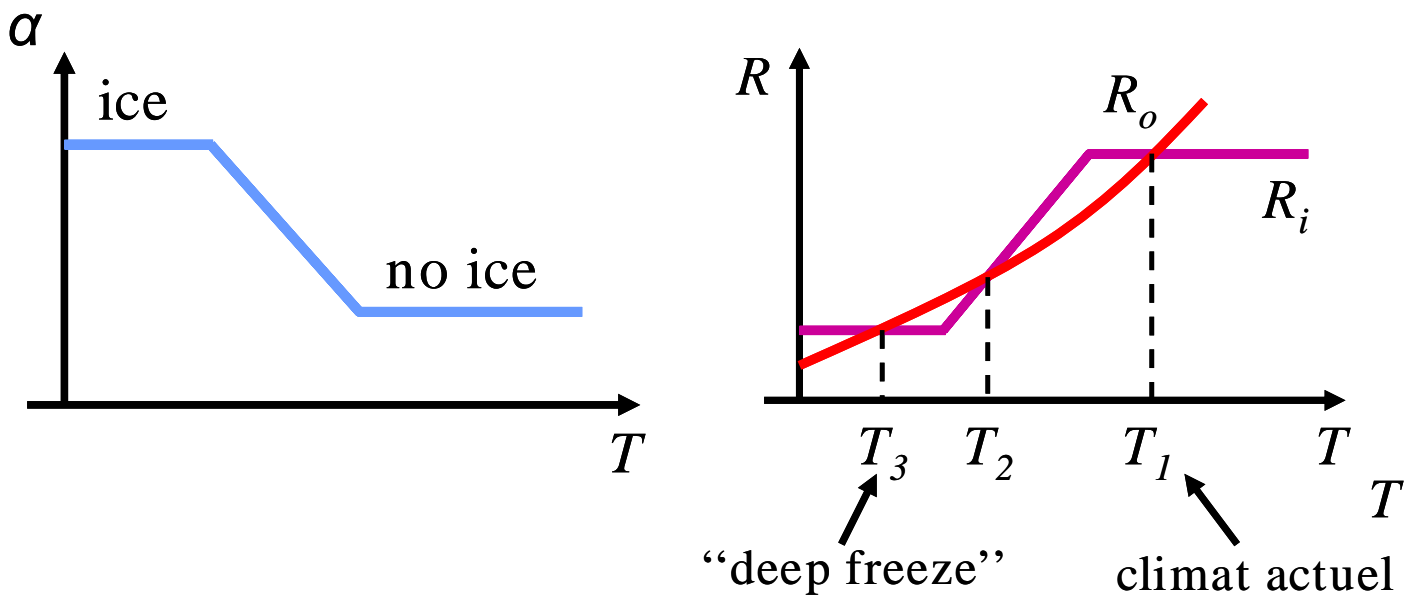
## Solutions du modèle

On veut écrire  $T$  sous la forme:

$$T = T(t; T_0, Q, c, \dots)$$

Solutions stationnaires :

$$Q \{1 - \alpha(T)\} - \sigma T^4 = 0$$



Que se passe-t-il si le soleil « clignote » et  $T = T_1 + \Delta T$ ?

Il faut reprendre l'équation d'origine, qui dépend du temps.

## Condition de stabilité

$$C \partial_t T = R_i - R_o = f(T)$$

$$R_i = Q \{1 - \alpha(T)\}$$

$$R_o = A + BT$$

On pose  $T = T_j + \theta$ :

$$f(T_j) = 0,$$

$$f(T) = f(T_j) + f'(T_j)\theta + \dots$$

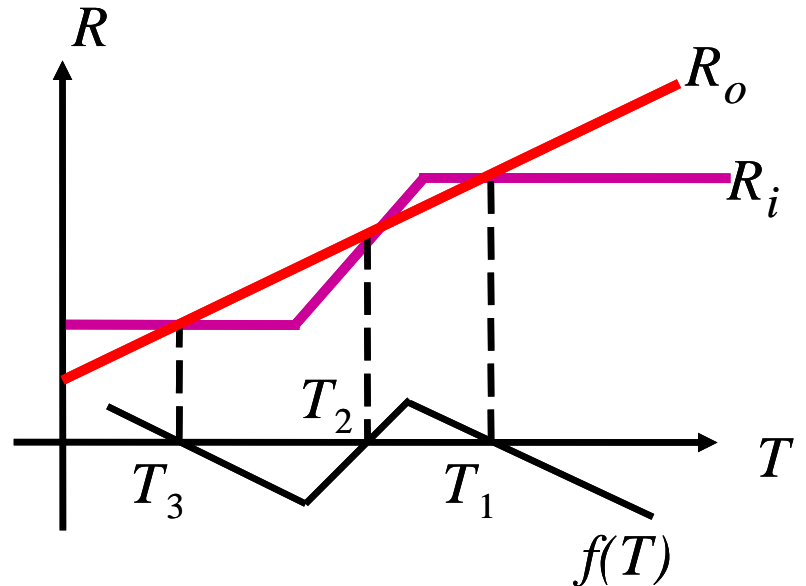
Soit  $\lambda_j \equiv f'(T_j)/c$

$$\partial_t \theta = \lambda_j \theta \Rightarrow \theta = e^{\lambda_j t} \theta_0.$$

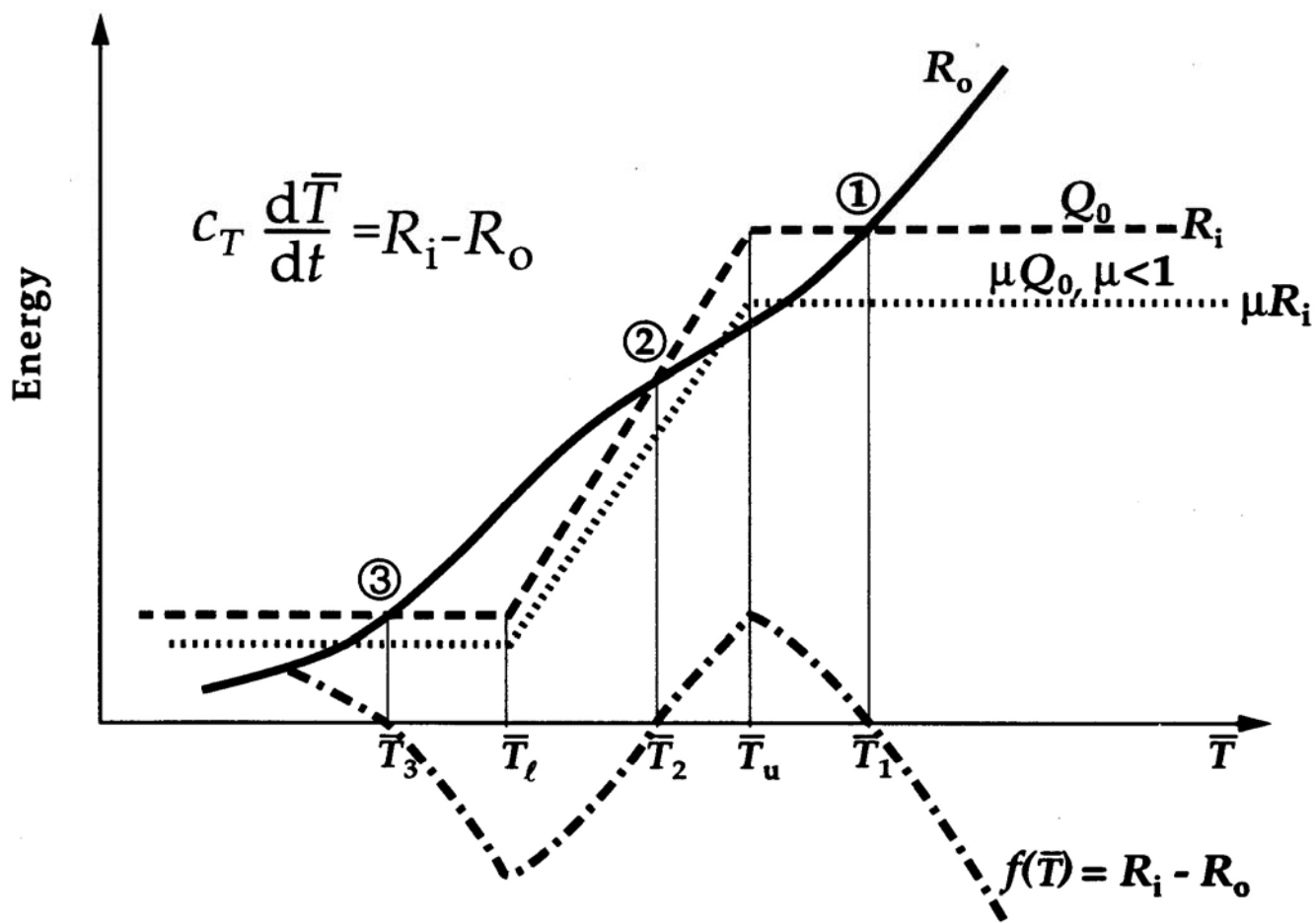
Si  $\lambda_j < 0$       stable

Si  $\lambda_j > 0$       instable

Remarque: dans le cas 1-D,  $\lambda_j \rightarrow \lambda_j^{(0)}$ ;  $\lambda_j \sim 1/c$







## Modèle 1-D (EBM "classique")

$$(1) C(x)T_t = R_i - R_o + D$$

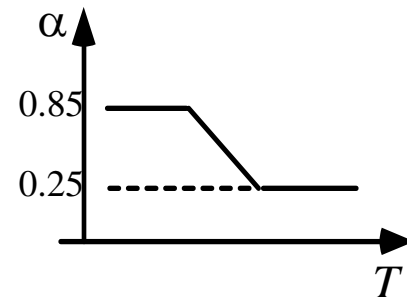
$T$  – température

$x$  – coordonnée latitudinale,  $0 \leq x \leq 1$

$$R_i = Q(x)\{1 - \alpha\}$$

$$= Q(x)\{1 - b(x) + c_1 T\}_c$$

$$R_o = \sigma T^4 \{1 - m \tanh(c_3 T^6)\}$$



$\tilde{T}(x)$  – le climat observé

$$D = \frac{1}{\sin \frac{\pi x}{2}} \partial_x \sin \frac{\pi x}{2} \{k(x) + k_s(x)g(\tilde{T})\} T_x$$

$$\text{C.B. : } T_x(0) = T_x(1) = 0$$

$x = 0$  Pôle (Nord)

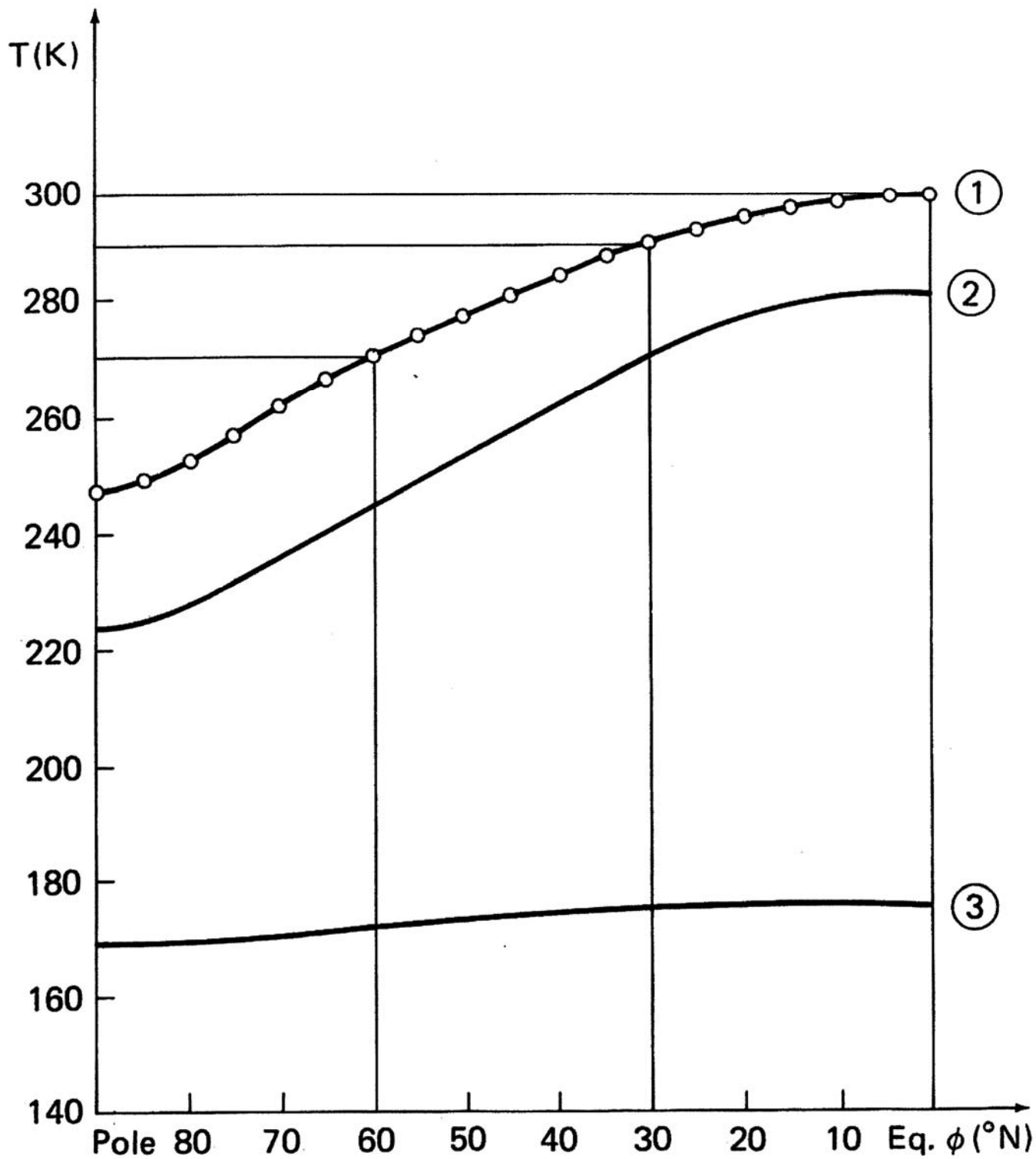
$x = 1$  Equateur

**Questions:** 1. Solutions stationnaires ("climats") ?

2. Stabilité ?

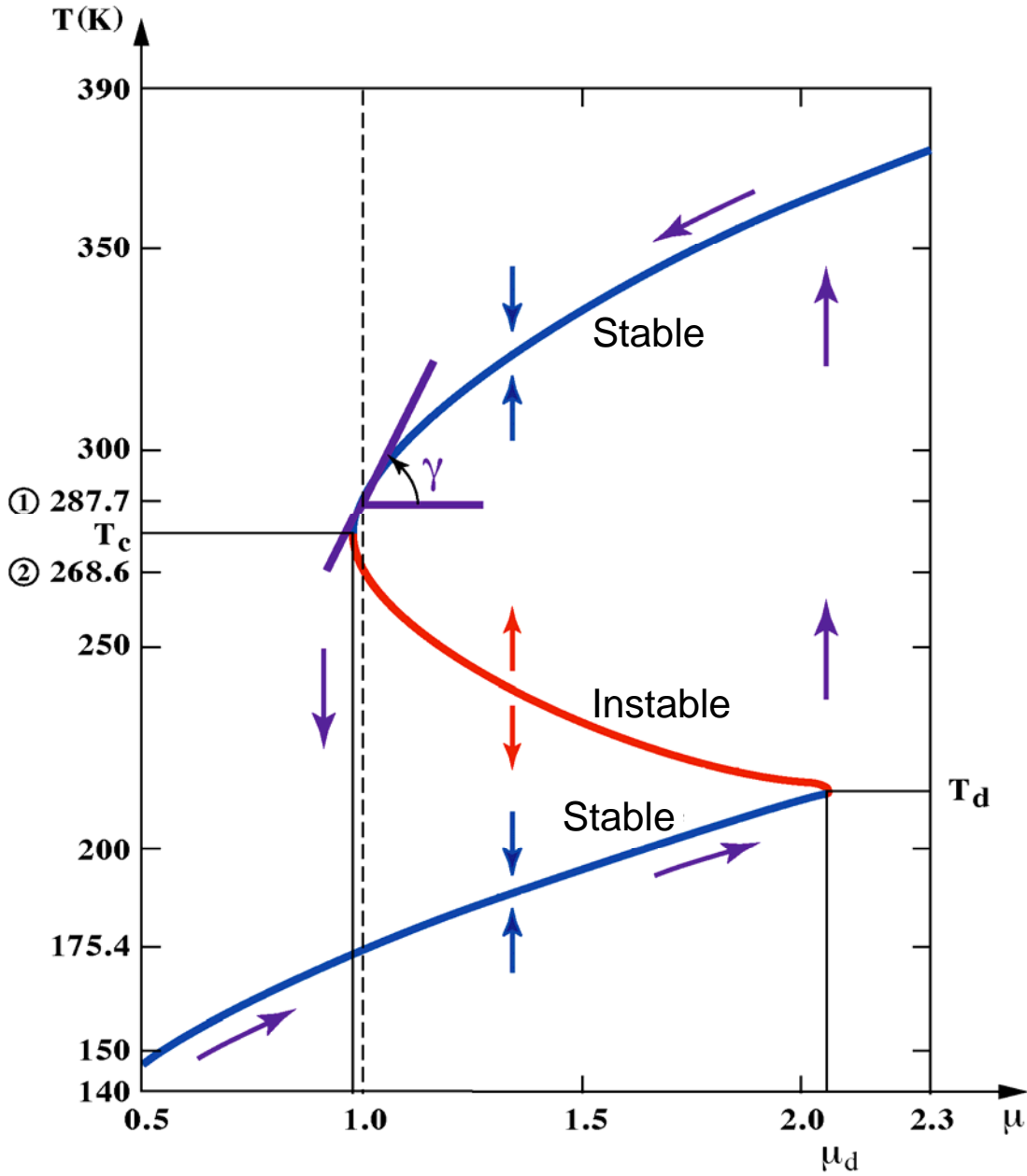
3. Perturbation & bifurcation ?

$$Q \rightarrow \mu Q \quad (\mu = 1)$$



# Diagramme de Bifurcation

**EBM 1-D:**  $C(x)T_t = \{k(x,T)T_x\}_x + \mu Q_0 \{1 - \alpha(x,T)\} - g(T)\sigma T^4$   
 $T_x = 0$  at  $x = 0,1$

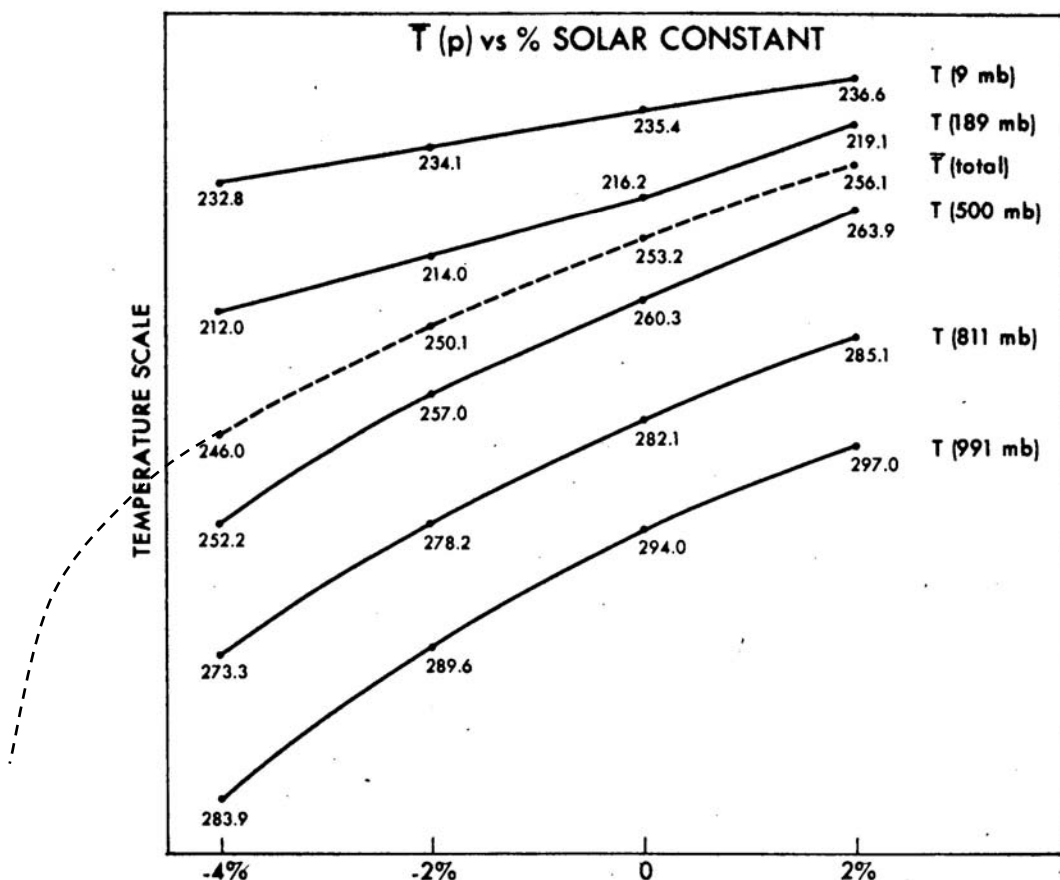


**Sensibilité du climat:**  $\gamma = \frac{dT}{d\mu} \cong 0,01$  (1K par % de  $Q$ )

# Sensibilité du climat à une variation de l'insolation dans un Modèle de Circulation Générale (MCG, GCM)

“As stated in the Introduction, it is not, however, reasonable to conclude that the present results are more reliable than the results from the one-dimensional studies mentioned above simply because our model treats the effect of transport explicitly rather than by parameterization.” \*

“Nevertheless, it seems to be significant that both the one-dimensional and three-dimensional models yields qualitatively similar results in many respects.” \*



Area-mean temperatures for various level of the model as well as a mass-weighted mean temperature for the total model atmosphere for each of the four cases. Vertical scale has been adjusted for each case separately and superimposed upon the others. Units are in K. \*

\* D'après Wetherald and Manabe (1975), *J. Atmos. Sci.*, 32, 2044-2059.

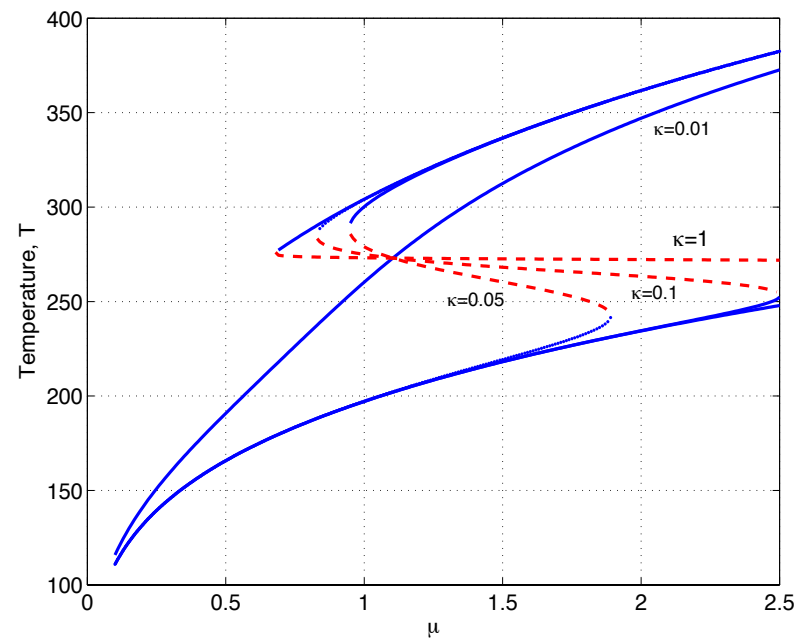
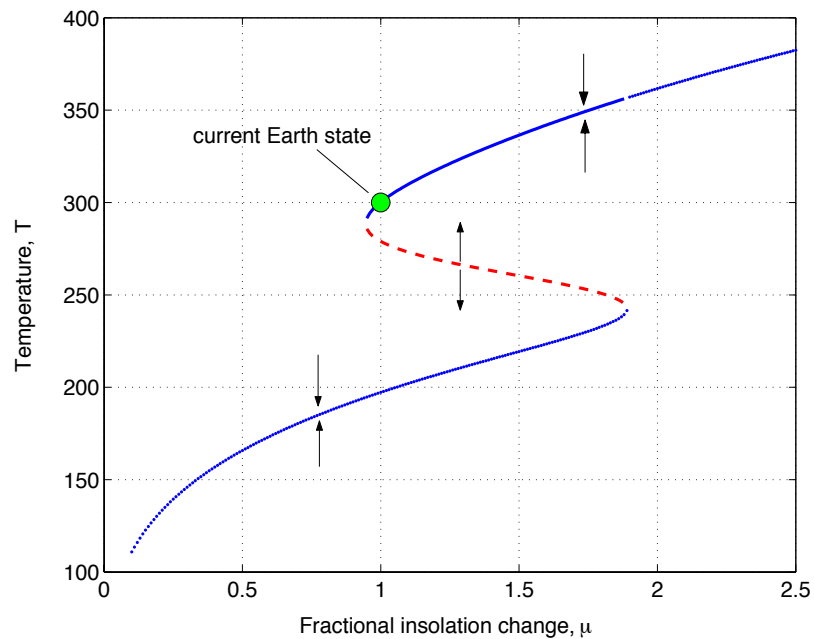
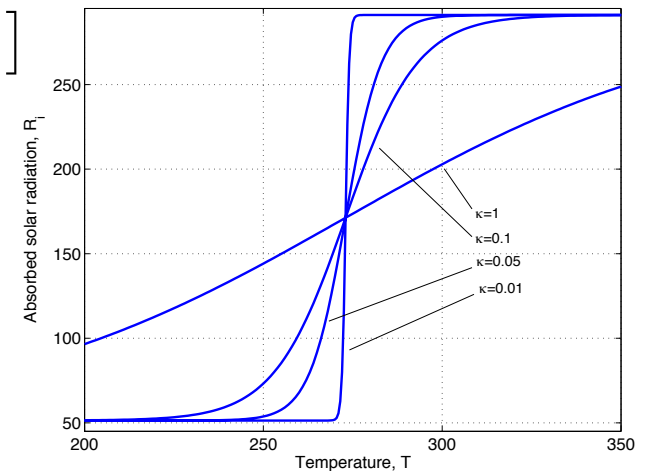
# Distance to “tipping points”?

Slightly modified 0-D EBM (Zaliapin & Ghil, *NPG*, 2010)

$$c\dot{T} = \mu Q_0 (1 - \alpha(T)) \sigma T^4 [1 - m \tanh((T/T_0)^6)]$$

$$\alpha(T; \kappa) = c_1 + c_2 \frac{1 - \tanh[\kappa(T - T_c)]}{2}$$

$T_c$  is the ice-margin temperature,  
while  $\kappa$  is an extra “Budyko-vs.-Sellers” parameter



# Double-well potential in 2-D

1-D EBM of Budyko-Sellers-North, cf. Held & Suarez (*Tellus*, 1974); North *et al.* (*JAS*, 1979).

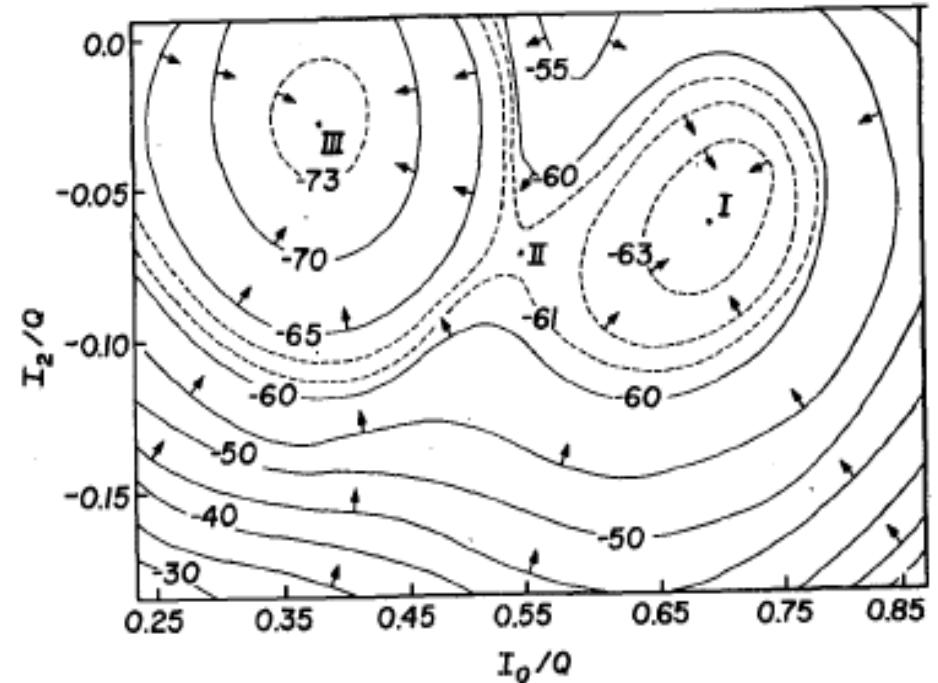
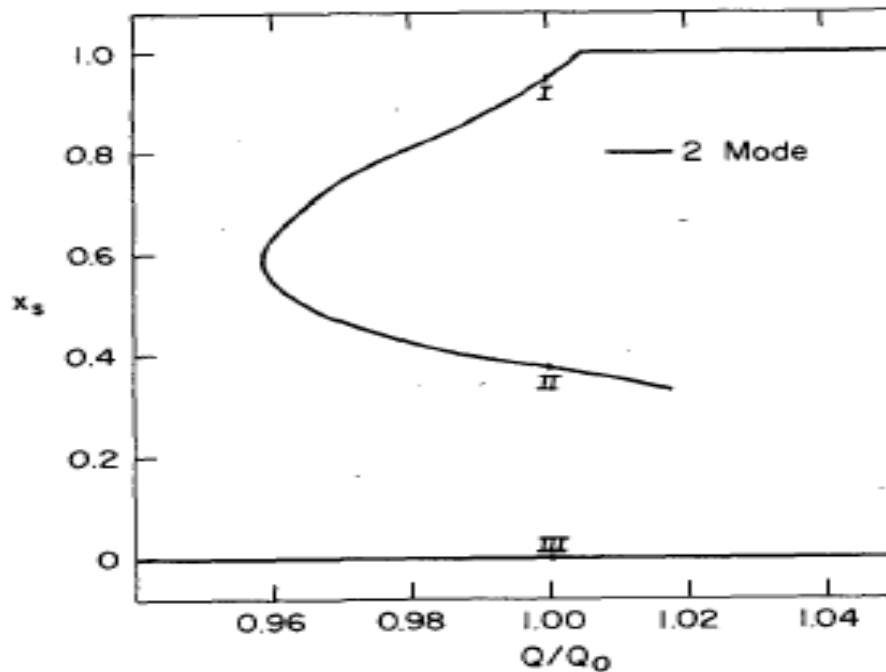
Taking  $x = \sin(\text{latitude})$  and  $k(x, T) = k_0$ ,  
we get the semi-linear parabolic PDE

$$CT_t = [k_0(1 - x^2)T_x]_x + Q(x)[1 - \alpha(T)] - I(T)$$

which yields the variational principle:

$$F\{T(x)\} = \int \left\{ \left[ \frac{1}{2} k_0 (1 - x^2) T_x^2 - Q(x) A(T) + R(T) \right] dx \right\}, \text{ where}$$

$$A(T) = \int^T [1 - \alpha(T)] dT, \text{ and } R(T) = \int^T I(T) dT.$$



# Concluding remarks, I

- ◆ Tipping points and bifurcations: multiple equilibria and rapid transitions between them.
- ◆ Prediction of the transitions? To follow.
- ◆ Transitions between more general types of behavior — limit cycles, strange attractors — likewise to follow.



# Some general references

- Andronov, A.A., and L.S. Pontryagin, 1937: Systèmes grossiers. *Dokl. Akad. Nauk. SSSR*, **14**(5), 247–250.
- Arnold, L., 1998: *Random Dynamical Systems*, Springer Monographs in Math., Springer, 625 pp.
- Arnol'd, V. I., 1983: *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York/Heidelberg/Berlin, 334 pp.
- Dijkstra, H.A., 2005: *Nonlinear Physical Oceanography : A Dynamical Systems Approach to the Large-Scale Ocean Circulation and El Niño*, 2nd edn., Springer, 532 pp.
- Dijkstra, H. A., and M. Ghil, 2005: Low-frequency variability of the large-scale ocean circulation: A dynamical systems approach, *Rev. Geophys.*, **43**, RG3002, doi:10.1029/2002RG000122.
- Ghil, M., R. Benzi, and G. Parisi (Eds.), 1985: *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, North-Holland, 449 pp.
- Ghil, M., and S. Childress, 1987: *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory and Climate Dynamics*, Ch. 5, Springer-Verlag, New York, 485 pp.
- Ghil, M., M.D. Chekroun, and E. Simonnet, 2008: Climate dynamics and fluid mechanics: Natural variability and related uncertainties, *Physica D*, **237**, 2111–2126.
- Lorenz, E.N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130–141.
- Ruelle, D., and F. Takens, 1971: On the nature of turbulence. *Commun. Math. Phys.*, **20**, 167–192.

**Reserve slides**

## Modèles de bilan énergétique (EBM) :

### Budyko, Sellers et Held-Suarez-North

Table 10.1. Comparison of Budyko's and Sellers' models.

Heat Flux	Budyko	Sellers
$R_i = Q(1 - \alpha(T))$ Absorbed solar radiation, as a function of ice-albedo feedback	Step-function albedo $\alpha = \begin{cases} \alpha_M, & T < T_s, \\ \alpha_m, & T \geq T_s, \end{cases}$ $\alpha_M > \alpha_m,$ $T_\ell \leq T_s \leq T_u$	Ramp-function albedo $\alpha = \begin{cases} \alpha_M, & T < T_\ell \\ \alpha_M - \frac{T - T_\ell}{T_u - T_\ell} (\alpha_M - \alpha_m), & T_\ell \leq T < T_u \\ \alpha_m, & T \geq T_u \end{cases},$
$R_0$ Outgoing IR radiation	Linear, empirical $A + BT$	Stefan-Boltzmann law with greenhouse effect $\sigma T^4 \{1 - m \tanh(T^6/T_0^6)\}$
$\nabla \cdot F$ Horizontal flux divergence	Newtonian cooling $\kappa\{T(\phi) - \bar{T}\}$	Eddy-diffusive $\nabla \cdot \{k(\phi) \nabla T(\phi)\}$

2e colonne : [Budyko \(1968, 1969\)](#) ; 3e colonne : [Sellers \(1969\)](#)

encadré en rouge : version "mixte"

(Held & Suarez, 1974; North, 1975a, b)